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ON AN EXTENSION OF THE THEOREM OF V. A. AMBARZUMYAN

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Introduction :

In a paper published in 1929, V. A. Ambarzumyan [1] proved the theorem that if $\{\lambda_n\}$, n = 0, 1, 2, ..., be the eigenvalues of the operator

$$y'' + (\lambda - q)\, y = 0,\ 0 \leq x \leq \pi,\ y'(0) = y'(\pi) = 0,$$

q (x) a real valued function of x continuous in $[0, \pi]$, and if $\lambda_n = n^2$, then $q = 0$. This theorem is considered as a first step towards the solution of the inverse problem associated with the Sturm—Liouville operator.

In the present paper we propose to extend this theorem to the matrix differential system

(1.1) $L\phi = \lambda\phi$

where $L \equiv \begin{pmatrix} -D^2+p & r \\ r & -D^2+q \end{pmatrix}$, $D \equiv d/dx$, p, q, r are real valued functions of x such that p, q, r are integrable over $[0, \pi]$ and $\phi = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \equiv \begin{pmatrix} u \\ v \end{pmatrix}$.

The boundary conditions to be satisfied by the solutions ϕ of (1.1) are

$$\text{(1.2)} \quad \left.\begin{array}{l} u'(0) = v'(0) = 0 \\ u'(\pi) = v'(\phi) = 0 \end{array}\right\},$$

the 'Neumann boundary conditions' at $x=0$ and $x=\pi$ respectively. The problem (1.1) along with (1.2) may be called the 'Neumann boundary value problem'.

When $p = q = r = 0$, the system (1.1) reduces to

(1.3) $$D^2 \phi + \lambda \phi = 0,$$

where $\phi = \begin{pmatrix} u\,(x) \\ v\,(x) \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$.

When the solutions of (1.3) satisfy the boundary conditions (1.2) we obtain the 'Fourier problem' corresponding to the 'Neumann boundary value problem' (1.1) and (1.2).

It is easy to verify that the eigenvalues for the Fourier problem are precisely given by the set $\{n^2\}$, $n=0, 1, 2, 3, \ldots$.

It is noted that equations of the form

$$Y'' + \lambda^2 Y = [v(x) + 6x^{-2}P]\, Y,\ 0<x<\infty$$

where $V = \| v_{jk}(x) \|_1^2$ is a Hermitian matrix satisfying

$$\int_0^\infty x^{1+\theta} \,|\, V(x) \,|\, dx < \infty,\ -\epsilon<\theta<\epsilon,\ 0<\epsilon<1.$$

and $P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ arises from the Schrödinger equation for a deuteron (in ground state) if tensor interaction forces are taken into account [Agranovich and Marchenko—The inverse problem in Scattering theory, Gordon Breach, NY, 1963, P.7].

2. Asymptotic estimates:

Let the solutions $\begin{pmatrix} u \\ v \end{pmatrix}$ of (1.1) satisfy the general boundary conditions at $x=0$ and $x=\pi$, viz.,

(2.1) $a_{j1}\, u\,(0) + a_{j2}\, u'\,(0) + a_{j3}\, v\,(0) + a_{j4}\, v'\,(0) = 0$

(2.2) $b_{j1}\, u\,(\pi) + b_{j2}\, u'\,(\pi) + b_{j3}\, v\,(\pi) + b_{j4}\, v'\,(\pi) = 0.$

$j=1, 2$, where a_{ij}, b_{ij} are real constants (independent of λ) such that

i) $\text{rank}\,(a_{ij}) = \text{rank}\,(b_{ij}) = 2$;

ii) $a_{j1}\, a_{k2} + a_{j3}\, a_{k4} = 0,\ j, k = 1, 2$;

iii) $b_{11}\, b_{22} - b_{12}\, b_{21} + b_{13}\, b_{24} - b_{14}\, b_{23} = 0$;

iv) $b_{j2}\, a_{k1} + b_{j4}\, a_{k3} = 0,\ j=1,2$;

v) $b_{j2}\, a_{12} + b_{j4}\, a_{14} \neq 0,\ \ b_{j1}\, a_{k2} + b_{j3}\, a_{k4} = 0\ ;\ j = 1, 2.$

Then the system (1.1) together with the boundary conditions (2.1), (2 2) and the conditions (j)—(iv) determines a self adioint eigenvalue problem in $[0, \pi]$.

Let $\{\lambda_n\}$, where $\lim_{n \to \infty} \lambda_n = \infty$, be the eigenvalues of the system (1.1) with the boundary conditions (2.1) and (2.2). Then to solve the present problem we exploit the analysis of Levitan and Gasymov (3, Appendix I, II) as follows.

Let A be the matrix $A = \begin{pmatrix} a_{12} & a_{14} \\ a_{11} & a_{13} \end{pmatrix}$ and let the ith row and the jth column of any matrix M be represented, respectively, by M_{i*} or $(M)_{i*}$ and M_{*j} or $(M)_{*j}$.

Put $X(x,t) \equiv \begin{pmatrix} X_1(x,t) \\ X_2(x,t) \end{pmatrix}$, $X(0,0)=0$, and $Y(x,t) \equiv \begin{pmatrix} Y_1(x,t) \\ Y_2(x,t) \end{pmatrix}$,

$Y(0,0)=0$, where X, Y have absolutely continuous partial derivatives with respect to x and t.

Then a necessary and sufficient condition that the vector

$$(2.3) \qquad \phi(x,\lambda) \equiv \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$$

$$= \begin{pmatrix} a_{12}\cos\sqrt{\lambda}\, x + a_{11}\sin\sqrt{\lambda}\, x \\ a_{14}\cos\sqrt{\lambda}\, x + a_{13}\sin\sqrt{\lambda}\, x \end{pmatrix} + \int_0^x \begin{pmatrix} X(x,t)\, S(t) \\ Y(x,t)\, S(t) \end{pmatrix} dt$$

where

$S(x) = \begin{pmatrix} \cos\sqrt{\lambda}\, x \\ \sin\sqrt{\lambda}\, x \end{pmatrix}$ and $X(x,t)S(t) = X_1(x,t)\cos\sqrt{\lambda}\, t + X_2(x,t)\sin\sqrt{\lambda}\, t$, with a similar meaning for $Y(x,t)\,S(t)$, is a solution of the given differential system (1.1) with boundary codditions (2.1) at $x=0$, is that all the condltions (3.6)–(3.9) of Ray Paladhi (4, Theorem 1, P. 172–175) are satisfied. One of the conditions explicitly required in our discussion is the following :

$$(2.4) \quad X'(x,x) = 1/2\, F_{*1}(x), \quad Y'(x,x) = 1/2\, F_{*2}(x),$$

where

$$F = \begin{pmatrix} a_{12}\,p + a_{14}\,r & a_{12}\,r + a_{14}\,q \\ a_{11}\,p + a_{13}\,r & a_{11}\,r + a_{13}\,q \end{pmatrix}.$$

Differentiation of (2.3) yields

$$(2.5) \quad \begin{pmatrix} u'(x) \\ v'(x) \end{pmatrix} = \sqrt{\lambda} \begin{pmatrix} a_{11}\cos\sqrt{\lambda}\, x - a_{12}\sin\sqrt{\lambda}\, x \\ a_{13}\cos\sqrt{\lambda}\, x - a_{14}\sin\sqrt{\lambda}\, x \end{pmatrix} + \begin{pmatrix} X(x,x)\, S(x) \\ Y(x,x)\, S(x) \end{pmatrix}$$

$$+ \int_0^x \begin{pmatrix} \frac{\partial}{\partial x} X(x,t)\, S(t) \\ \frac{\partial}{\partial x} Y(x,t)\, S(t) \end{pmatrix} dt.$$

When $\lambda = \lambda_n$ is an eigenvalue for the problem (1.1) with (2.1) and (2.2) so that $\phi(x,\lambda_n)$ may now represent the corresponding eigenvector, it follows from (2.2) on substitution for $\begin{pmatrix} u \\ v \end{pmatrix}$, $\begin{pmatrix} u' \\ v' \end{pmatrix}$ as given by (2.3) and (2.5), utilization of the relation (iv) satisfied by a_{ij}, b_{ij} and subsequent reductions, that for $j=1,2,$

(2.6) $$\left[(B_j)^T_{2*} A^T_{1*} + (B_j)^T_{1*} \Omega_{*1}(\pi,\pi)\right] \cos\sqrt{\lambda_n}\,\pi$$
$$+\left[(B_j)^T_{2*} A^T_{2*} - \sqrt{\lambda_n}(B_j)^T_{1*} A^T_{1*} + (B_j)^T_{1*} \Omega_{*2}(\pi,\pi)\right] \sin\sqrt{\lambda_n}\,\pi$$
$$+\int_0^\pi \left\{(B_j)^T_{2*} \Omega_{*1}(\pi,t) + \frac{\partial}{\partial x}(B_j)^T_{1*} \Omega_{*1}(x,t)\Big|_{x=\pi}\right\} \cos\sqrt{\lambda_n}\,t\,dt$$
$$+\int_0^\pi \left\{(B_j)^T_{2*} \Omega_{*2}(\pi,t) + \frac{\partial}{\partial x}(B_j)^T_{1*} \Omega_{*2}(x,t)\Big|_{x=\pi}\right\} \sin\sqrt{\lambda_n}\,t\,dt = 0,$$

where $B_j = \begin{pmatrix} b_{j2} & b_{j4} \\ b_{j1} & b_{j3} \end{pmatrix}$, $\Omega(x,t) = \begin{pmatrix} X_1(x,t) & X_2(x,t) \\ Y_1(x,t) & Y_2(x,t) \end{pmatrix}$

and A, A_{1*}, A_{*1}, etc. are as defined before. $\alpha\beta$ as usual stands for

$\alpha_1\alpha_2 + \beta_1\beta_2$ where $\alpha = \begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}, \beta = \begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix}$.

From (2.6), as $n \to \infty$

(2.7) $$\sin\sqrt{\lambda_n}\,\pi + o(\lambda_n^{-\frac{1}{2}}) = 0$$

Also by adopting the analysis of Titchmarsh (5, P. 19), it is easy to deduce that $\lambda_n \sim n^2$. as n tends to infinity [see for example Chakravarty—Q. J. M 19(74), 1968, P. 216)]. So that a first approximation of λ_n is given by

$\sqrt{\lambda_n} = n + o(1/n)$, as n tends to infinity.

Put

$\sqrt{\lambda_n} = n + a_j/n + \Upsilon_n/n$, j = 1,2, where a_j are constants independent of n and Υ_n tends to zero as n tends to infinity, implying that

$$\sqrt{\lambda_n} \sim n + a_j/n.$$

Then

(2.8) $$\sin\sqrt{\lambda_n}\,\pi = (-1)^n \pi\,(a_j + \Upsilon_n)/n + o(n^{-3})$$
$$\text{and } \cos\sqrt{\lambda_n}\,\pi = (-1)^n(1 + o(n^{-2}).$$

Also by the Riemann—Lebesgue theorem, the integral terms in (2.6) vanish as n and therefore λ_n tends to infinity. Hence from (2.6) by using the relations (2.8), we have

$$a_j = \left((B_j)^T_{2*} A^T_{1*} + (B_j)^T_{1*} \Omega_{*1}(\pi,\pi)\right) / [\pi (B_j)^T_{1*} A^T_{1*}]$$

$\left((B_j)^T_{1*} A^T_{1*} \neq 0\text{, by the condition (v) on } a_{ij}, b_{ij}.\right)$

Since p, q, r are integrable over $[0,\pi]$ (or in particular p,q,r are continuous over $[0,\pi]$), it follows by using the relation (2.4) that

$$(2.9)\quad a_j=\left[(B_j)^T_{2*} A^T_{1*} + 1/2 \int_0^{\pi} (B_j)^T_{1*} F^T_{1*}(t)dt\right] \Big/ \left[(\pi \; (B_j)^T_{1*} A^T_{1*} \right], \quad j=1,2.$$

The vector with two components a_1, a_2 so obtained may be called the 'boundary characteristic vector' of the given problem. In particular, a_1 may be equal to a_2.

3. Solution of the problem :

If, $(B_j)^T_{2*} A^T_{1*}=0$, and if the elements of $(B_j)^T_{1*}$ as well as the constants a_{12}, a_{14} assume values independent of each other, it follows from (2.9) that

$$(3.1)\quad \int_0^{\pi} p\,dx=\int_0^{\pi} q\,dx=\int_0^{\pi} r\,dx=0.$$

when $a_j=0$.

Let the eigenvalues for the Neumann boundary value problem be given by $\{n^2\}$, n $=0,1,2,\ldots$. Then the vector (a_1,a_2) is null. Also for the Neumann boundary conditions, the requirements relating to $(B_j)^T$, A^T, etc as stated above are satisfied. Therefore in this case (3.1) holds.

Now let c_n be the Fourier coefficient of a vector $f=\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, such that $f \in C^1(I), I=[0,\pi]$, and f′ is absolutely continuous over I, If $f'(0)=f'(\pi)=0$, then from Chakravarty and Sen Gupta (2, formula (3.2) P.23),

$$(3.2)\quad D(f,f)=D(f) \geqq \sum_{n=0}^{\infty} \lambda_n c_n^2 \geqq \lambda_0 \sum_{n=0}^{\infty} c_n^2=\lambda_0 \, \| f \|^2$$

where

$$D(f)=\int_0^{\pi}\left\{ | f' |^2 + f^T P f \right\} dx, \quad P=\begin{pmatrix} p & r \\ r & q \end{pmatrix}, \quad \lambda_n \geqq \lambda_0 \geqq 0.$$

The equality in (3.2) holds if and only if f is an eigenvector corresponding to the eigenvalue λ_0 for the Neumann boundary value problem over $[0, \pi]$ and now

$$(3.3)\quad \lambda_0=\min\left(D(f,f)/ \|f\|^2 \right)$$

the minimum being taken over all $f \neq 0 \epsilon D$, satisfying the Neumann boundary conditions at $x=0$ and $x=\pi$, where D is the set of all complex-valued vector functions continuous in $[0, \pi]$ and having piecewise derivatives in the same interval.

Let $f_0 = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$ be an arbitrarily chosen non-null constant vector.

f_0 satisfies the Neumann boundary conditions at $x=0$ and $x=\pi$. Also by utilizing (3.1), it follows that $D(f_0)=0$. Hence from (3.3), f_0 is the eigenvector corresponding to the minimum eigenvalue '0' of the sequence $\{n^2\}$, $n=0, 1, 2, 3, \ldots$. Therefore from (1.1)

$$pC_1 + rC_2 = 0, \qquad rC_1 + qC_2 = 0,$$

leading to $p=q=r=0$ almost everywhere in $[0, \pi]$.

We thus obtain the following theorem.

Theorem : A necessary and sufficient condition that the system (1.1) with the boundary conditions (1.2) reduces to the corresponding Fourier problem (i.e., the system (1.3) with boundary conditions (1.2)) is that the eigenvalues of the given system are characterised by $\{n^2\}$, $n=0, 1, 2, 3, \ldots$.

Remark

An elaborate and revised version of the present paper is due to appear in the Proceedings of the Royal Society of Edinburgh, where some of the shortcomings and ambiguities have been corrected.

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