Jour. Pure Math.
Vol. 5 (1985—86), pp. 7–14

ON THE RIESZ SUMMABILITY OF A FOURIER TYPE EXPANSION OF A TWO COMPONENT FUNCTION

N. K Chakravarty

Abstract

A theorem on Riesz Means of order $l \geqslant 0$ for a Fourier type series or the F—series of a two component function $(f_1, f_2)^T$ differentiated p times, is obtained under a set of conditions analogous to that of Fe'jer—Lebesgue in ordinary Fourier Series.

1. Introduction : The theorem.

Cousider the differential system

(1.1) $$d^2u/dx^2 + \lambda u = 0$$
$$d^2v/dx^2 + \lambda v = 0$$

in the interval $[0, \pi]$, where λ is the eigenvalue parameter.

Let the solutions $(u, v)^T$ of (1.1) satisfy the following boundary conditions :

At $x=0$,

(1.2) $a_{j1}\, u(0) + a_{j2}\, u'(0) + a_{j3}\, v(0) + a_{j4}\, v'(0) = 0$, $j = 1,2$, where

i) rank $(a_{ij}) = 2$, $i = 1,2$, $j = 1,2,3,4$;

ii) $a_{j1}\, a_{k2} + a_{j3}\, a_{k4} = 0$, $j,k = 1,2$;

iii) $(a_{jn}, a_{jm}) \neq (0,0)$, when $j=1$, $n=1$, $m=3$ and when $j=2$, $n=2$, $m=4$;

and at $x=\pi$,

(1.3) $b_{j1}\, u(\pi) + b_{j2}\, u'(\pi) + b_{j3}\, v(\pi) + b_{j4}\, v'(\pi) = 0$, $j = 1,2$,

iv) $b_{11}\, b_{22} - b_{12}\, b_{21} + b_{13}\, b_{24} - b_{14}\, b_{23} = 0$;

v) rank $(b_{ij}) = 2$, $i = 1,2$, $j = 1,2,3,4$.

a_{ij}, b_{ij} are real valued constants independent of the parameter λ.

The eigenvalue problem associated with the system (1.1) along with the boundary conditions (1.2) and (1.3) over the function space $L_2(0, \pi)$ is well-known. (See e. g. Chakravarty [2], p.135 – 150). The problem is self-adjoint by the conditions (ii), (iv) on a_{ij}, b_{ij}. We call this problem, the E—problem.

Let $f(x)=(f_1,f_2)^T$ possess continuous derivatives upto the order two ; or more generally, let f (x) be the integral of an absolutely continuous function on $(0, \pi)$. If f (x) satisfy the boundary conditions (1.2), (1.3), respectively, at $x=0$, $x=\pi$, then f (x) admits of the eigenfunction expansion

(1.4) $$f(x) = \sum_{0}^{\infty} C_n \Psi_n(x),\ 0 \leqslant x \leqslant \pi,$$

where $C_n = \int_0^\pi (f, \Psi_n)\, dt$, are the Fourier coefficients of f ; Ψ_n (x) are the normalized eigenvectors corresponding to the eigenvalue λ_n of the E—problem. The series is uniformly and absolutely convergent for $0 \leqslant x \leqslant \pi$.

Let a_{ij} , b_{ij} satisfy additional conditions

(1.5) $b_{km_1} a_{jm_2} + b_{kn_1} a_{jn_2} = \alpha_{kj}$, k, j=1, 2, ($b_{ij}$ not all zero), where $\alpha_{kj}=0$, if (m_1, m_2, n_1, n_2) are the arrangements (1, 2, 3, 4), (2, 1, 4, 3) of (1, 2, 3, 4), $\alpha_{kj} \neq 0$ for other arrangements. In view of the conditions (1.2)—(ii), (iii) on a_{ij}, it follows from (1.5) that $(b_{jn}, b_{jm}) \neq (0,0)$, when j=1, n=1, m=3, and when j=2, n=2, m=4.

Put $D_j = \frac{1}{2} \pi^{\frac{3}{2}} D_0^{-\frac{1}{2}} | a_j |^2$, j=1, 2,

and $D_0 = \frac{3}{8}\pi^3 | a_1 |^2 | a_2 |^2 | a_1 - a_2 |^2 > 0$

where $a_j = (a_{j1}, a_{j2}, a_{j3}, a_{j4})$ with usual norm $| a_j |^2$ and inner product (a_1, a_2).

If

(1.6) $$C_j(x,n) = \begin{pmatrix} a_{j2} \cos nx + a_{j1} \sin nx \\ a_{j4} \cos nx + a_{j3} \sin nx \end{pmatrix}$$

where n is a positive integer, then it can be easily verified that

(1.7) $$\Psi_n(x) = \left\{ D_1 C_2(x, n) - D_2 C_1(x, n) \right\} / \pi^{\frac{1}{2}}$$

is a normalized eigenvector corresponding to the eigenvalue n^2 associated with the E-problem. (Compare Acharyya [1], where eigenvectors in this form occur in a different context.)

Definition. The series on the right hand side of (1.4) where Ψ_n (x) has the explicit representation (1.7) is defined as the "Fourier type series corresponding to the two component function f (x)", or shortly, the F-series of our problem.

Evidently, the F-series is an eigenfunction expansion of a two component function f (x) (with none of its components a classical Fourier series) when f satisfies certain conditions including boundary conditions as above stated. F-series as an eigenfunction expansion in used only for mean convergence considerations needed in the context.

It may be noted that the choice of $\Psi_n(x)$ need not be restricted to the form (1.7). In fact, if

$K_J(x, n)= \begin{pmatrix} A_J \cos nx+B_J \sin nx \\ C_J \cos nx+D_J \sin nx \end{pmatrix}$ and A_J, B_J, C_J, D_J are suitably restricted, it is possible to choose a linear combination of $K_1(x, n)$, $K_2(x, n)$ and a normalizing constant, not so simple as that in (1.7), as the normalized eigenvector for the E-problem.

The summability problems (including Riesz summability) for the Fourier series of scalar functions have been extensively studied ; but no such problems involving the F-series appear to have been taken up. The standard methods available for the single component functions cannot also be readily extended to hold in the present case.

Levitan and Sargsyan ([5] p. 54—57) obatined for the ordinary Fourier series $F\{f(x)\}$ of $f(x)$ a theorem on the summability of $F'\{f(x)\}$ at $x=x_0$ by Riesz means of order one to $f'(x_0)$, when (i) $f'(x)$ exists and is continuous at x_0 and (ii) $f'(x)$ is integrable in the neighbourhood of x_0. Their method is considerably different from the available ones. They use in their investigation the Tauberian theorem :

Theorem A. Let $\sigma(v)$ be a function of bounded variation in every finite interval, such that

$$\overset{\mu+1}{\underset{\mu}{V}} \sigma(\nu)=o(|\mu|^r),\ r>0,\ \int_{-\infty}^{\infty} h(\nu)\, d\sigma(\nu)=0, \text{ where}$$

$h(\nu)=\frac{1}{2\pi}\int_{-\epsilon}^{\epsilon} K_\epsilon(t) \exp(-i\nu t)\, dt$, K_ϵ is an arbitrary function having bounded $(r+2)$ th derivative and K_ϵ vanishes outside $(-\epsilon, \epsilon)$. Then for all $s \geqslant 0$,

$\int_{-\infty}^{\infty}(1-\nu^2/\mu^2)^s\, d\sigma= o(|\mu|^{r-s})$, as μ tends to infinity, the passage to the limit being uniform. (See Levitan and Sargsyan ([5], p. 85, Appendix)).

In the present note we investigate the problem involving Riesz means of order $l \geqslant 0$ for a p-times differentiated F-series under a set of conditions satisfied by $f(x) \equiv (f_1, f_2)^T$ analogous to those of Fe'jer—Lebesgue for the classical Fourier Series. We note that a series $(\Sigma a_n, \Sigma b_n)^T$ is summable (R, λ, k) if Σa_n, Σb_n are so in the usual sense. (See Chandrasekharan and Minakshisundaram [4]). The theorem to be proved is stated as follows.

Theorem 1. 1. Let $f(x)$ be a two component function, p times differentiable on $(0, \pi)$, such that

$$\int_0^t | f^p(x+u) - f^p(x) |^r \, du = o(t^r), \; r \geq 1, \tag{1.8}$$

as t tends to zero, where $x \in (x_0 - \delta, x_0 + \delta)$, x_0 fixed and $\delta > 0$.

Then

(1.8a) (i) $$\lim_{\mu \to \infty} \int_0^\mu (1 - \nu^2/\mu^2)^l \, d_\nu \, S^{(p)}(x, \nu) = f^{(p)}(x) + o(\mu^{p-l}) :$$

the result holds uniformly for $x_0 - \delta \leq x \leq x_0 + \delta$, $\delta > 0$; $S(x, \mu)$ is given by

$$S(x,\mu) = \sum_{k<\mu} C_k \Psi_k(x) \tag{1.9}$$

ii) The p-times differentiated F-series is summable at x_0 by the Riesz means of order l to $f^{(p)}(x_0)$, $l \geq p \geq 0$, the result holding uniformly for $x_0 - \delta \leq x \leq x_0 + \delta$, $\delta > 0$.

iii) In particular, if $f^{(p)}(x) \in L_r(0, \pi)$, $r \geq 1$, the the result is valid almost everywhere in $(0, \pi)$.

The second part follows from the first part from definition. The third part is an immediate consequence of the second partand the Minkowsky inequality, since (1.8) holds for almost all x, if $f^{(p)} \in L_r(0, \pi)$, $r \geq 1$. (See Zygmund [7], p. 237). It is therefore enough to establish the first part of the theorem which we do in the following. We follow Levitan and Sargsyan [3] indicating steps but emphasizing the parts where we considerably differ.

2. Proof of the theorem. Let $\{f_n(x)\}$ be a sequence of vectors defined on $(0, \pi)$ satisfying the conditions of validity of the expansion formula (1.4) when $\Psi_n(x)$ is given by (1.7). Then

$$f_n(x) = \sum_{k=0}^{\infty} C_k^{(n)} \Psi_k(x), \quad C_k^{(n)} = \int_0^\pi (f_n, \Psi_k) \, dt. \tag{2.1}$$

the series being absolutely and uniformly convergent for $0 \leq x \leq \pi$.

Let $\{f_n(x)\}$ converge to $f(x)$ in the norm of $L(0, \pi)$.

As in Levitan and Sargsyan ([5], p. 20-22) let $g_\epsilon(x)$ be a scalar function, satisfying

i) $g_\epsilon(x) = g_\epsilon(-x)$, (ii) $g_\epsilon(x) = 0$ for $|x| \geq \epsilon$ and (iii) $g_\epsilon(x)$ has a bounded second derivative. If $\phi_\epsilon(\mu$ is the Fourier cosine transform of $g_\epsilon(x)$:

(2.2) $\phi_\epsilon(\mu) = \int_0^\epsilon g_\epsilon(u) \cos \mu u \, du,$

then $\phi(\mu)$ is even and

(2.3) $\phi(\mu) = o(1/\mu^2)$, as μ tends to infinity, by integration by parts.

It essily follows from (1.6) and (1,7) that

(2.4) $\frac{1}{2}[\Psi_n(x+t) + \Psi_n(x-t)] = \Psi_n(x) \cos nt$

Then from (2.1)—(2.4),

(2.5) $$\int_0^\epsilon [f_n(x+t) + f_n(x-t)] \, g_\epsilon(t) \, dt = \int_{-\infty}^{\infty} \phi_\epsilon(\mu) \, d_\mu S_n(x, \mu)$$

where

$$S_n(x, \mu) = \sum_{k<\mu} C_k^{(n)} \Psi_k(x), \; n=1, 2, \ldots$$

The mean convergence consideration permits us to pass on to the limit under the sign of integration on the left hand side of (2.5) and the same operation is permissible on the right hand side of (2.5) in view of (2.3). Therefore from (2.5)

(2.6) $$\int_{-\infty}^{\infty} \phi_\epsilon(\mu) \, d_\mu S(x, \mu) = \int_0^\epsilon [f(x+t) + f(x-t)] \, g_\epsilon(t) \, dt$$

where $S(x, \mu)$ is given by (1.9).

In view of the finiteness of $S(x, \mu)$ and the relation (2.3), it is possible to differentiate the left hand side of (2.6) with respect to x under the sign of integration and the same process is obviously applicable to the right hand side. Hence from (2.6),

(2.7) $$\int_{-\infty}^{\infty} \phi_\epsilon(\mu) \, d_\mu S^{(p)}(x, \mu) = \int_0^\epsilon [f^{(p)}(x+t) + f^{(p)}(x-t)] \, g_\epsilon(t) \, dt$$

where $x \in (x_0 - \frac{1}{2}\delta, \; x_0 + \frac{1}{2}\delta)$ and the superscript p indicates the pth derivative of the concerned function.

Put

(2.8) $$\alpha(x, \mu) = (\alpha_1, \alpha_2)^T = \int_0^\epsilon [f^{(p)}(x+t) + f^{(p)}(x-t)] \cos \mu t \, dt$$

Then applying the Parseval theorem for ordinary Fourier Transform to each component of $\alpha(x, \mu)$ and (2.2), it follows that

$$(2.9) \quad \int_{-\infty}^{\infty} \phi_{\epsilon}(\mu)\, d_{\mu}[S^{(p)}(x,\mu) - 1/\pi \int_{0}^{\mu} \alpha(x,\nu)\, d\nu] = 0.$$

since $\phi_{\epsilon}(\mu)$ and $\alpha(x,\mu)$ are each even functions of μ.

It can be easily verified that

$$\underset{\mu}{\overset{\mu+1}{V}} [S^{(p)}(x,\mu)] = O(\mu^{p}), \text{ as } \mu \text{ tends to infinity,}$$

also $\underset{\mu}{\overset{\mu+1}{V}} \left[\int_{0}^{\mu} \alpha(x,\nu)\, d\nu \right] = o(1)$, as μ tends to infinity.

Hence from (2.9) and the Tauberian theorem A,

$$(2.10) \quad \int_{0}^{\mu} (1-\nu^2/\mu^2)^{l}\, d_{\nu}\, [\overset{9}{S}{}^{(p)}(x,\nu) - \frac{1}{\pi} \int_{0}^{\nu} \alpha(x,u)\, du\,] = o\,(\mu^{p-l})$$

as μ tends to infinity uniformly for $x_0 - \frac{1}{2}\delta \leqslant x \leqslant x_0 + \frac{1}{2}\delta$.

By a change in the order of integration, which is easily justifiable, we have

$$(2.11) \quad \pi \int_{0}^{\mu} (1-\nu^2/\mu^2)^{l}\, d_{\nu}\, S^{(p)}(x,\nu)$$

$$= \int_{0}^{\epsilon} \phi(t)\, dt \int_{0}^{\mu} (1-\nu^2/\mu^2)^{l} \cos \nu t\, d\nu$$

$$+ 2f^{(p)}(x) \int_{0}^{\epsilon} dt \int_{0}^{\mu} (1-\nu^2/\mu^2)^{l} \cos \nu t\, d\nu + o\,(\mu^{p-l})$$

$$= I_1 + I_2 + o(\mu^{p-l}), \quad \text{as } \mu \text{ tends to infinity,}$$

where $\phi(t) = f^{(p)}(x+t) + f^{(p)}(x-t) - 2f^{(p)}(x)$.

In I_2, (let us evaluate the inner integral by Watson [6] p.48, formula (3)), change the the variable in the intergral and then utilize the Weber integral (Watson [6], P. 39).

By fixing ϵ and making μ tend to infinity, it follows that I_2 tends to $f^{(\nu)}(x)$ as μ tends to infinity.

It follows from (1.8) that

$$(2.12)\quad \Phi(t) = \int_0^t |\phi(u)|^r du < \epsilon_1 t^r$$

where ϵ_1 is a preassigned positive quantity, t being small enough.
Evaluating the inner integral in I_1 and then changing μt to t, we have

$$(2.13)\quad I_1 = 2^{l-\frac{1}{2}} \Gamma(l+\tfrac{1}{2}) \pi^{\frac{1}{2}} \int_0^{\mu\epsilon} \phi(t/\mu) J_{l+\frac{1}{2}}(t)/t^{l+\frac{1}{2}} dt$$

The integral on the right hand side of (2.13) is equal to

$$A = \left[\int_0^1 + \int_1^{\mu} + \int_{\mu}^{\mu\epsilon}\right] (.)\, dt$$
$$= I_{11} + I_{12} + I_{13}, \text{ say.}$$

Then

$$|A|^r \leqslant 3^{r-1}(|I_{11}|^r + |I_{12}|^r + |I_{13}|^r), r \geqslant 1.$$

Using the inequality

$$|J_\nu(z)/z| < 1/2^\nu \Gamma(\nu+1), \nu > -\tfrac{1}{2}, \quad \text{(Watson [6], p.49)}$$

and the inequality

$$(2.14)\quad |F|^r \leqslant (b-a)^{r-1} \int_a^b |f|^r dt$$

for a vector

$F = \int_0^b f\, dt$, we have $|I_{11}|^r < \epsilon_1/\mu^{r-1}$.

Since $|J_\nu(z)| \leqslant B$, for any real value of $z > 1$, where B is a constant, it easily follows that

$$|I_{13}|^r \leqslant B^r/\mu^{r(l-\frac{1}{2})} (\epsilon-1)^{r-1} (\Phi(\epsilon) - \Phi(1)).$$

To estimate I_{12}, we have, by integration by parts,

$$| I_{12} |^{r} \leqslant H(\mu) [\Phi(1) - \mu^{r(l+\frac{1}{2})} \Phi(1/\mu) + r(l+\tfrac{1}{2}) \int_{1/\mu}^{1} t^{-r(l+\frac{1}{2})-1} \Phi(t) \, dt]$$

$= J_{11} + J_{12} + J_{13}$, say,

where $H(\mu) = B^r/\mu^{r(l-\frac{1}{2})}(1-1/\mu)^{r-l}$ tends to zero, as μ tends to infinity, if $r \geqslant 1$, $l > \frac{1}{2}$,

Now J_{11} tends to zero, as μ tends to infinity, since $H(\mu)$ does so. Also, by (2.12), $| J_{12} | < \epsilon_1$ as μ tends to infinity.

To estimate J_{13}, we divide the interval of integration $(1/\mu, 1)$ into sub intervals $(1/\mu, \eta)$ and $(\eta, 1)$ and choose η such that for $0 \leqslant t \leqslant \mu$, $\Phi(t) < \epsilon_1 t^r$. Then by familiar arguments $| J_{13} | < \epsilon_1$, as μ tends to infinity, uniformly in $x_0 - \delta \leqslant x \geqslant x_0 + \delta$, $\delta > 0$.

Altogether, from (2.13), $I_1 = o(1)$, as μ tends to infinity, uniformly in $x_0 - \delta \leqslant x \leqslant x_0 + \delta$, $\delta > 0$. Hence from (2.11) we obtain (1.8a), valid for $l \geqslant 0$, $p \geqslant 0$.

When $r=1$ in (1,8), (1.8a) follows by an easy adaptation of Hobson ([7], p.567—569) by replacing the function $C_{l+k}(t)$ by the Bessel function $J_{l+\frac{1}{2}}(t)$.

The theorem in thus established.

The author expresses his grateful thanks to Dr. D. K. Ganguli of the Dept. of Pure Mathematics, Calcutta University for helpful and constructive criticisms of the manuscript.

REFERENCES

[1] Acharyya, Sudip Kumar — Some eigenvalue problems associated with a matrix differential operator— Calcutta Ph.D. thesis (1983)

[2] Chakravarty, N. K. — Some problems in eigenfunction expansions I, Quart J. Math (Oxford) (2) 16 (62) 1965, 135-150.

[3] Chandrasekharan, C. and Minakshisundaram, S.— Typical Means (1952) (Oxford University Press).

[4] Hobson, E. W. — Theory of function of a real variable, Vol. II (Dover) 1926.

[5] Levitan, B. M. and Sargsyan, I. S. — Some problems in the theory of a Sturm-Liouville equation— Russian Math Surveys, Vol 15 (1960) p. 2-95.

[6] Watson, G. N. — Bessel Functions (1944) (Cambridge Univ, Press).

[7] Zygmund, A.— Trigonometrical Series (Dover) (1955).

Received (Revised form)
25. 3. 1985

Department of Pure Math
Calcutta University