Jour. Pure Math.
Vol. 5 (1985—86), pp. 37—44

A NOTE ON PAIRWISE SEMIOPEN SETS IN A SUBSPACE OF A BITOPOLOGICAL SPACE

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ABSTRACT In this paper, pairwise semiopen and semiclosed sets and their different properties, especially with regard to subspaces of a bitopological space, have been studied.

Norman Levine [2] introduced semiopen and semiclosed sets in a topological space and investigated some of their properties. This paved the way for a very wide research in this direction. The corresponding notions were introduced in the more generalized and richer structure of bitopological spaces by S. Bose [1]. It is the purpose of this paper to make a further study of the concepts of pairwise semiopen, semiclosed sets and pairwise semiclosure and semiinterior of sets of a bitopological space with special reference to subspaces. Such a study turns out to be of immense help and sometimes indispensible when study of certain semitopological concepts e. g. semiconnectedness in bitopological space, is carried out in terms of subspace topology. To make the expositions self-contained as far as practicable, we quote a few definitions and results from [1] as follows. By (X, T_1, T_2) or simply by X we shall mean a bitopological space. A subset A of (X, T_1, T_2) will be called (1, 2)-semiopen [1] in X, written as (1,2)-SO(X), if there exists a T_1 open set B such that $B \subset A \subset T_2$-clB (T_2-clB denotes the T_2-closure of B in X). Similarly, sets which are (2, 1)-semiopen in X are defined. A ($\subset$X) is called (i, j)-semiclosed (in short, (i, j)-Scl (X)) if X-A is ij -SO (X) where i, j =1, 2 and $i \neq j$. The set of all subsets that are (i, j)-semiopen (semiclosed) in X will be denoted by (T_i , T_j)-SO (X) (respectively by (T_i , T_j)-SCl (X)), for i, j =1, 2 and $i \neq j$. The subset A of X is called pairwise semiopen or simply p. s. o. (pairwise semi-closed or simply p. s. cl.) in X [1] if A is (1, 2)-SO (X) and (2,1)-SO (X) (respectively, (1, 2)-SCl (X) and (2, 1)-SCl (X)).

It has been shown in [1] that a set in (X, T_1, T_2) may be semiopen in both (X, T_1) and (X, T_2) but is neither $(1,2)$-SO(X) nor is $(2,1)$-SO(X). Also a set may be p. s. o. without being either T_1 or T_2 semiopen. Obviously every T_i-open set in (X, T_1, T_2) is ij-SO(X) for $i, j = 1, 2$ and $i \neq j$, but the converse is false. Though the union of any collection of sets, each of which is (i, j)-SO(X), is also so, the intersection of even two sets that are (i, j)-SO(X), may not be (i, j)-SO(X) $(i, j = 1, 2$ and $i \neq j)$. This is seen from the following example. :

EXAMPLE 1 Let $X = \{a, b, c, d\}$, $T_1 = \left\{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\right\}$ and $T_2 = \left\{X, \emptyset, \{a\}, \{b, d\}, \{a, c\}, \{a, b, d\}\right\}$. Then (T_1, T_2)–SO$(X)=$

$$\left\{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}, \{a, c\}, \{b, c, d\}\right\} \text{ and } (T_2, T_1)\text{–SO}(X)=$$

$$\left\{X, \emptyset, \{a\}, \{b, d\}, \{a, c\}, \{a, b, d\}, \{a, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\right\}.$$

Now, $\{b, c\}$, $\{a, c\}$ are $(1, 2)$–SO(X) but $\{b, c\} \cap \{a, c\} = \{c\}$ is not so, though $\{b, c\} \in T_1$. Again $\{a, c, d\}$ and $\{b, c, d\}$ are $(2, 1)$–SO(X) but their interesection i. e., $\{c, d\}$ is not $(2, 1)$–SO(X).

Theorem 1 In a bitopological space (X, T_1, T_2) if A is (1,2)-SO(X) and $B \in T_1 \cap T_2$ then $A \cap B$ is (1, 2)-SO(X).

Proof. There is T_1-open set V such that $V \subset A \subset T_2\text{-cl}V$. Since B is T_2-open, we have $T_2\text{-cl } V \cap B \subset T_2\text{-cl } (V \cap B)$.

Thus $V \cap B \subset A \cap B \subset T_2\text{-cl}V \cap B \subset T_2\text{-cl } (V \cap B)$. Since $V \cap B$ is T_1-open, it now follows that $A \cap B$ is (1, 2)-SO(X).

Remark 1. It is clear that in the above theorem if A be (2, 1)-SO(X), then $A \cap B$ is also so.

Definition 1 (a) Let x be a point of (X, T_1, T_2). A subset N of X is called a (1, 2)-semi-neighbourhood of x [1] in X if there is a (1, 2)-SO(X) set B (say) such that $x \in B \subset N$. Similarly, (2, 1)-semi-neighbourhood of x in X is defined.

Definition 1 (b) Let $A \subset (X, T_1, T_2)$. A point x of X is said to be a (1, 2)-semi accumulation point of A in X if every (1, 2)-semi-neighbourhood of x intersects A in at least one point other than X. Similar goes the definition of (2, 1)-semi accumulation point of A in X.

Definition 1 (c) Let $A \subset (X, T_1, T_2)$. The intersection of all (i, j)-SCl (X) sets, each containing A, is called the (i, j)-semi-closure of A in X [1] and is denoted by $\underline{A}_{T_i(T_j)}$, where i, j=1, 2 and i ≠ j.

Therem 2 (Bose [1]) Let $A \subset (X, T_1, T_2)$. (a) A is (i, j)-SCl(X) if and only if $A = \underline{A}_{T_i(T_j)}$,

(b) $x \in \underline{A}_{T_i(T_j)}$ if and only if x is either a point of A or a (i, j)-semi accumulation point of A in **X**. In (a) and (b), i, j=1, 2 and i ≠ j.

From Theorem 2, we have

Theorem 3 A set A in (X, T_1, T_2) is (i, j)-SCl(X) if and only if A contains the set of all (i, j)-semi accumulation points of A in X (i, j=1, 2 and i ≠ j),

Theorem 4 Let $Y \subset (X, T_1, T_2)$, Then a subset U of Y is (1, 2)- SO(Y) $\Rightarrow U = V \cap Y$, for some (1, 2)-SO(X) set V (here (1, 2)-SO(Y) means U is (1, 2)-semiopen in the space $(Y, (T_1)_Y, (T_2)_Y)$.

Proof. $U(\subset Y)$ is (1, 2)-SO(Y)

$\Rightarrow$ there is a $W \in (T_1)_Y$ such that $W \subset U \subset (T_2)_Y\text{-cl}W$,

But, $W = V_1 \cap Y$, where $V_1 \in T_1$.

Now, $U = (V_1 \cup (U - W)) \cap Y$ and $V_1 \cup (U - W) \subset T_2\text{-cl}V_1$.

In fact, $U - W \subset U = U \cap Y \subset (T_2)_Y\text{-cl}W \cap Y \subset T_2\,\text{cl}W \cap Y$

$= T_2\text{-cl}(V_1 \cap Y) \cap Y \subset T_2\text{-cl}\, V_1 \cap T_2\text{-cl}\, Y \cap Y \subset T_2\text{-cl}\, V_1 \cap Y \subset T_2\text{-cl}\, V_1$,

Thus putting $V_1 \cup (U - W) = V$ we see that $U = V \cap Y$,

where $V_1 \subset V \subset T_2\text{cl}\, V_1$ and $V_1 \in T_1$, i,e., V is (1, 2)-SO(X).

Remark 2 Converse of Theorem 4 is false even if Y be T_1-open: as is seen from the example below.

Example 2. Consider (X, T_1, T_2) of Example 1 and Y={ a, c, d }.

We have $(T_1)_Y = \{Y, \emptyset, \{a\}, \{c\}, \{a, c\}\}$,

$(T_2)_Y = \{Y, \emptyset, \{a\}, \{d\}, \{a, c\}. \{a, d\}\}$

$((T_1)_Y, (T_2)_Y) - SO(Y) = \{Y, \emptyset, \{a\}, \{c\}, \{a, c\}\}$

$((T_2)_Y\ (T_1)_Y)-SO(Y)=\left\{Y, \emptyset, \{a\}, \{d\}, \{a, c\}, \{a, d\}\right\}$

Now, $\{b, c, d\} \in (T_1,T_2)-SO(X)$ but $\{b, c, d\} \cap Y=\{c, d\} \notin \left((T_1)_Y, (T_2)_Y\right)-SO(Y)$.

Also, $\{b, c, d\} \in (T_2,T_1)-SO(X)$ but $\{b, c, d\} \cap Y=\{c, d\} \notin$

$\left((T_2)_Y, (T_1)_Y\right)-SO(Y)$.

Again, $Z=\{a, b, d\}$ is T_2—open.

Then, $(T_1)_z=\left\{\{Z, \phi, \{a\}, \{b\}, \{a, b\}\right\}$

$(T_2)_z=\left\{Z, \phi, \{a\}, \{b,d\}\right\}$.

$\left((T_1)_z, (T_2)_z\right)-SO(Z)=\left\{Z, \phi, \{a\}, \{b\}, \{a, b\}, \{b, d\}\right\}$.

$\left((T_2)_z, (T_1)_z\right)-SO(Z)=\left\{Z, \phi, \{a\}, \{b,d\}, \{a, d\}\right\}$.

Now, $\{a, b, c\} \in (T_2,T_1)-SO(X)$, but $\{a, b, c\} \cap Z$

$=\{a, b\} \notin \left((T_2)_z, (T_1)_z\right)-SO(Z)$, though $Z \in T_2$.

Theorem 5 Let $Y \subset (X, T_1, T_2)$. A subset F of Y is $(1, 2)$—SCI $(Y) \Rightarrow F=V \cap Y$, for some $(1, 2)$.—SCI (X) set V.

Proof. We have $Y-F$ is $(1, 2)$—SO(Y), Then by Theorem 4, $Y-F=Y \cap O$, where O is $(1. 2)$—SO(X). Then $F = Y-(Y \cap O)=(X-O) \cap Y$, where $V=(X-O)$ is $(1, 2)$—SC! (X).

Remark 3 The converse of Theorem 5 is false even if Y is both T_1—closed and T_2—closed. This is shown in the next example

Example 3 Let $(X\ T_1, T_2)$ be same as in Example I, and let $Y=\{b, c, d\}$.

Y is T_1—closed as well as T_2—closed.

We have $(T_1)_Y=\left\{Y, \phi, \{b, c\}\right\}$, $(T_2)_Y=\left\{Y, \phi, \{c\}, \{b, d\}\right\}$,

$\left((T_1)_Y, (T_2)_Y\right)-SO(Y)=\left\{Y, \phi, \{b, c\},\right\}$.

$\left((T_2)_Y, (T_1)_Y\right)-SO(Y)=\left\{Y, \phi, \{c\}, \{b, d\}, \{b, c\} \{c, d\}\right\}$.

Now, $\{b, d\}$ is $(1, 2)$—SCI(X), but $\{b, d\} \cap Y=\{b, d\}$ is not $(1, 2)$—SCI(Y), although Y is T_1—closed and T_2—closed.

From Theorem 5, it immediately follows that

Theorem 6 Let $A \subset Y \subset (X, T_1, T_2)$.

(a) If $x\ (\in Y)$ is a (i, j)-semi-accumulation point of A in X then x is also a (i, j)-semi-accumulation point of A in $(Y, (T_i)_Y, (T_j)_Y)$.

(b) $\underline{A}_{T_i(T_j)} \cap Y \subset \underline{A}_{(T_i)_Y((T_j)_Y)}$

In (a) and (b) above, $i, j = 1, 2$ and $i \neq j$.

Remark 4 The reverse inclusion in Theorem 6(b) does not, in general, hold as is seen from the next example.

Example 4 Consider (X, T_1, T_2) and Y of Exampla 2. Let $A = \{a\} \subset Y$. Then

$\underline{A}_{(T_1)_Y((T_2)_Y)} = \{a, d\}$, $\underline{A}_{(T_2)_Y((T_1)_Y)} = \{a, c\}$.

But $\underline{A}_{T_1(T_2)} = \underline{A}_{T_2(T_1)} = \{a\}$.

Theorem 7 Let $A \subset Y \subset (X, T_1, T_2)$. If $Y \in T_j$, then $\underline{A}_{T_i(T_j)} \cap Y = \underline{A}_{(T_i)_Y((T_j)_Y)}$, where $i, j = 1, 2$ and $i \neq j$.

Proof. We prove the theorem by taking $i = 1$ and $j = 2$. Similar will be the proof when $i = 2$ and $j = 1$.

Let us put $\underline{A}_{(T_1)_Y((T_2)_Y)} = A_1$

By virtue of theorem 6, it is enough to show that $A_1 \subset \underline{A}_{T_1(T_2)} \cap Y$, where $Y \in T_2$. Let $x \in A_1$ and B be $(1, 2)$-SO(X) such that $x \in B$. Then there is a T_1-open set O such that $O \subset B \subset T_2\text{-cl}\ O$.

Then $O \cap Y \in (T_1)_Y$ and $O \cap Y \subset B \cap Y \subset T_2\text{-cl}\ O \cap Y \subset (T_2)_Y\text{cl}(O \cap Y)$ (since $Y \in T_2$), Thus $B \cap Y$ is $(1,2)$-semi neighbourhood of x in Y. Since $x \in A_1$, by Theorem 2 (b) $A \cap (B \cap Y) \neq \emptyset$ and hence $A \cap B \neq \emptyset$, i.e., $x \in \underline{A}_{T_1(T_2)}$ Thus $A_1 \subset \underline{A}_{T_1(T_2)}$ and this completes the proof.

Remark 5 Theorem 7 may not hoid if we replace the hypothesis "$Y \in T_j$" by "$Y \in T_i$" or by "$Y \in (T_j, T_i)$-SO (X)".

This is verified in the following example.

Example : 5 We consider (X, T_1, T_2) and Z of Example 2.

Let $A = \{d\} \subset Z$. Then $\underline{A}_{T_2(T_1)} = \{d\}$ and $\underline{A}_{(T_2)_Z((T_1)_Z)} = \{b, d\}$.

Thus $\underset{-}{A}_{T_2(T_1)} \cap Z \neq \underset{-}{A}_{(T_2)_Z((T_1)_Z)}$, though $Z \in T_2$.

Also Y of Example 2 is (2,1)−SO (X) and if
$A=\{a\} \subset Y$, then as shown In Example 4,

$$\underset{-}{A}_{T_2(T_1)} \cap Y \neq \underset{-}{A}_{(T_2)_Y((T_1)_Y)}.$$

Theorem 8 (Bose [1]) Let $A \subset Y \subset (X,T_1,T_2)$. Then A is (i,j)−SO(X)⇒A is (i,j)−SO(Y), where i,j=1,2 and i ≠ j.

Remark 6 Converse of Theorem 8 is false as is shown by the example that follows.

Example 6 Consider the bitopological space (X,T_1,T_2) and the subset Y of Example 2. Then $\{c\}$ ($\subset Y$) is (1,2)—SO(Y) but is not (1,2)—SO(X) and $\{d\}$ is (2,1)-SO(Y) but is not (2,1)—SO(X).

Theorem 9 Let $A \subset Y \subset (X,T_1,T_2)$. If $Y \in T_1$, then A is (1,2)—SO (Y) if and only if A is (1,2)—SO(X).

Proof. Let A be (1,2)-SO(Y). Then there exists $(T_1)_Y$- open set O such that

$O \subset A \subset (T_2)_Y\text{-cl } O$. Since Y Is T_1-open, we have $O \in T_1$ and also $O \subset A \subset (T_2)_Y\text{-cl } O$

$\subset T_2\text{-cl } O$. Hence A is (1,2)-SO(X).

The other part follows from Theorem 8.

Remark 7 It is evident that the indices 1 and 2 can be interchanged in Theorem 9.

Remark 8 Theorem 9 does not hold, in general, if we replace condition "$Y \in T_1$" by "$Y \in T_2$" or by "$Y \in$ (1,2)-SO(X)". This is evident from the next example.

Example 7 Let us consider the bitopological space (X,T_1,T_2) and the subset Y of X of Example 2. We see that $Y \in$ (2,1)-SO(X) but $\{d\} \in ((T_2)_Y, (T_1)_Y)$-SO(Y), whereas $\{d\} \notin (T_2,T_1)$-SO(X). Again Z of the same Example 2 is T_2-open. Now $\{b\}$ is (1,2)-SO(Z) but $\{b\}$ is not (1,2)-SO(X).

Remark 9 Let $A \subset Y \subset (X,T_1,T_2)$. Then A is (1,2)-SCl(Y) does not imply nor is implied by the fact that A is (1,2)-SCl(X). In fact, we have the following.

Example 8 Consider Y and (X,T_1,T_2) of Example 2. Here $\{a\}$ ($\subset Y$) is (1,2)-SCl(X) but it is not (1,2)-SC1(Y).

Again, $\{c,d\}$ ($\subset Y$) is (1,2)-SC1(Y) but is not (1,2)-SC1(X).

Theorem 10 Let $A \subset Y \subset (X, T_1,T_2)$. If A is (i,j)-SC1(Y) and Y is(i,j)-SC1(X), then A is(i, j)-SC1(X), where i,j=1,2 and i ≠j.

Proof. follows easily from Theorem 5 and the fact that intersection of two (i, j)-SC1(X)-sets is also so.

Remark 10 A subset A of Y $\subset (X, T_1, T_2)$ may be (1,2)-SC1(X) but may not be (1,2)-SCI(Y), even if Y is pairwise semiopen, closed in both T_1 and T_2 and pairwise semi closed.

This is seen in the example that follows.

Example 9 Consider Y and (X, T_1, T_2) of Example 3. Then Y is pairwise semi open, pairwise semi closed and closed in both T_1, T_2. Now {b,c} is (2, 1)-SC1 (X) but it is not (2, 1)-SCI(Y).

Remark 11 It follows from Example 9 that if A is (i, j)-SC1(X) and $Y \subset X$, then $A \cap Y$ may not be (i, j)-SCI(Y) (i, j=1,2 ; i $\neq$ j), even if Y is both T_1 and T_2-closed (and hence pairwise semi closed).

Definition 2 [1] Let $A \subset (X, T_1, T_2)$. The union of all (1, 2)-SO(X) sets, each contained in A, is called the (1, 2)-semi interior of A in X and is denoted by (T_1, T_2)-SInt(A). Similarly (T_2, T_1)-SInt (A) is defined. It is proved in [1] that a set $A \subset (X, T_1, T_2)$ is (i, j)-SO(X) if A=(T_i, T_j)-SInt(A), where i, j=1, 2 and i $\neq$ j.

Theorem 11 Let $A \subset Y \subset (X, T_1, T_2)$. Then

(a) (T_1, T_2)-SInt(A) $\subset ((T_1)_Y, (T_2)_Y)$-SInt (A), where the reverse inclusion does not hold, in general,

(b) (T_1, T_2)-SInt (A) = $((T_1)_Y, (T_2)_Y)$-SInt(A) if Y is T_1-open.

Proof. (a) By virtue of Theorem 8, obviously (T_1, T_2)-SInt(A) $\subset ((T_1)_Y, (T_2)_Y)$-SInt(A).

Now, let us consider (X, T_1, T_2) and Y of Example 2.

Let A={c, d}. Then (T_1, T_2)-SInt (A)=(T_2, T_1)-SInt (A)=$\emptyset$.

But $((T_1)_Y, (T_2)_Y)$-SInt (A)={c} and $((T_2)_Y, (T_1)_Y)$ SInt (A)={d}.

(b) follows from theorem 9.

Remark 12 The converse of Theorem 11 (b) does not hold, in general. For example, if X is the real line and $T_1=T_2$=the usual topology and Y=[0, 1), then the equality in (b) holds for every $A \subset Y$ but clearly Y is neither T_1 open nor T_2 open,

Theorem 12 In a bitopological space (X, T_1, T_2), $T_i=S_i$, where $S_i=\{T_i\text{-int } A : A \in (T_i, T_j)\text{-SO}(X)\}$ (i,j=1, 2 and i $\neq$ j).

Proof. It is simple.

Acknowledgement I must thank Dr. S. Ganguly, Reader, Department of Pure Mathematics, Calcutta University, for his kind help and suggestion in the preparation of this paper.

Note This paper was communicated in early '82 for publication in Math. Balkanica. In early '84, the Editor of the Journal asked me to revise the paper in the light of the suggestion of the learned referee to whom I am extremely thankful. Subsequently a revised copy was sent in March '84. Since the paper which was supposed to have been published by now, has not appeared, the author has decided to publish an abbridged version of the paper in the Departmental Journal.

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Received
5.8. 1985

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