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GROUP-THEORETIC ORIGINS OF CERTAIN GENERATING FUNCTIONS FOR MODIFIED HYPERGEOMETRIC POLYNOMIALS—I

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1. INTRODUCTION :

In 1982, we [1] have discussed a unified theory of generating functions for hypergeometric polynomials $F(-n, \beta ; \gamma ; x)$ by giving suitable interpretations to n, β and γ simultaneously. The object of the present paper is to investigate the modified hypergeometric polynomials by giving a suitable interpretation to the index n only and we have observed that some generating functions obtained for $F(-n, \beta ; \gamma ; x)$ by means of triple interpretation are derived with little labour. Moreover, some new generating functions which do not appear in the investigation of the polynomial $F(-n, \beta ; \gamma ; x)$, are obtained. The main results are listed below :

$$(1.1) \quad (1-t)^n F\left(-n, \beta ; \gamma-n ; \frac{x-t}{1-t}\right) = \sum_{k=0}^{n} \frac{(-n)_k (\gamma-\beta-n)_k}{(\gamma-n)_k \, k!} F(-n+k, \beta ; \gamma-n+k ; x)\, t^k$$

which is due to Das [1, (6.5)] replacing γ by $\gamma+n$ in (1.1),

$$(1.2) \quad (1-y)^{\gamma-n-1}(1-xy)^{-\beta} F\left(-n, \beta ; \gamma-n ; \frac{1-y}{1-xy} . x\right) = \sum_{k=0}^{\infty} \frac{(1-\gamma+n)_k}{k!} F(-n-k, \beta ; \gamma-n-k ; x) y^k ,$$

which is again due to Das [1, (6.6)] replacing Υ by $\Upsilon+n$ in (1.2)

$$(1.3) \quad (1-y)^{\gamma-1-n}(1-xy)^{-\beta}(1-y+wy)^n F\left[-n, \beta ; \Upsilon-n ; \left(1+\frac{wxy}{1-xy}\right).\right.$$

$$\cdot \frac{1-y}{1-y+xy}\Big] = \sum_{k=0}^{\infty} \frac{y^k}{k!} \sum_{m=0}^{n} \frac{(-n)_m(\Upsilon-n-\beta)_m}{(\Upsilon-n)_m\, m!}(1-\Upsilon-m+n)_k.$$

$$(wy)^{n-m} \;.\; F(m-n-k, \beta; \Upsilon-k+m; x)$$

(1.4) $$(wy-1)^n \left(\frac{1+w-wxy}{w}\right)^{-\beta}\left(\frac{1+w-wy}{w}\right)^{y-1-n} F\Big[-n, \beta; \gamma-n; \frac{1-wxy}{1-wy}$$

$$\cdot \frac{1+w-wy}{1+w-wxy}\Big] = \sum_{k=0}^{\infty} \frac{(wy)^{n-k}}{k!} \sum_{m=0}^{k-n} \frac{(1-\gamma+n)_m(-n-m)_k(\gamma-n-m-\beta)_k}{(\gamma-n-m)_k\; m!} y^m.$$

$$.\; F(k-n-m, \beta;\; \gamma-m+k-n; x).$$

(1.5) $$e^{-y}\, {}_1F_1(\beta; \beta-\gamma+1; y-xy)$$

$$= \sum_{n=0}^{\infty} \frac{(1-\gamma)_n(-y)^n}{(\beta-\gamma+1)_n\, n!} F(-n, \beta; \gamma-n; x),$$

which do not seem to appear before.

(1.6) $$(1-y)^{\gamma-1}(1-xy)^{-\beta} \exp\left(\frac{-wy}{1-y}\right) {}_1F_1\left[\beta; \beta-\gamma+1; \frac{(1-x)wy}{(1-y)(1-xy)}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(1-\gamma)_n}{(\beta-\gamma+1)_n} F(-n, \beta; \gamma-n; x)\; L_n^{\beta-\gamma}(w)y^n,$$

which is a new bilateral generating relation parallel to a result of L. Weisner [2 ; (4.6)].

(1.7) $$x^{-\beta}e^{-t}\, {}_1F_1\left(\beta; \beta-\gamma+1; \frac{x-1}{x}\, t\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(1-\gamma)_n}{(\beta-\gamma+1)_n\, n!} F(-n, \beta; \gamma-n; x)(-t)^n,$$

where $x \neq 0$ and this does not appear before.

2. LINEAR DIFFERENTIAL OPERATORS :

Modified hypergeometric polynomials $F(-n, \beta; \Upsilon-n; x)$ satisfy the differential equation

(2.1) $$x(1-x)\frac{d^2y}{dx^2} + [\gamma-n-(\beta-n+1)x]\frac{dv}{dx} + n\beta = 0.$$

Replacing $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$, n by $y\frac{\partial}{\partial y}$ and y by u (x, y), we obtain from (2.1),

$$(2.2)\quad x(1-x)\frac{\partial^2 u}{\partial x^2}-(1-x)\,y\frac{\partial^2 u}{\partial x\partial y}+(\gamma-(\beta+1)\,x)\frac{\partial u}{\partial x}+\beta y\frac{\partial u}{\partial y}=0,$$

of which $u=y^n F(-n, \beta ; \gamma-n ; x)$ is a solution, since $F(-n, \beta ; \gamma-n ; x)$ is a solution of (2.1).

As we know that $F(-n, \beta ; \gamma-n ; x)$ satisfies the following

$$(1-x)\frac{d}{dx}F(-n, \beta ; \gamma-n ; x)$$

$$=\frac{n(\gamma-\beta-n)}{\gamma-n}F(-n+1, \beta ; \gamma-n+1 ; x)-n\,F(-n, \beta ; \gamma-n ; x)$$

and

$$x(1-x)\frac{d}{dx}F(-n, \beta ; \gamma-n ; x)$$

$$=(\gamma-n-1)\,F(-n-1, \beta ; \gamma-n-1 ; x)-[\gamma-n-1-(\beta-2n-1)\,x]\,F(-n, \beta ; \gamma-n ; x)$$

we find two partial differential operators

$$(2.3)\quad B=(1-x)y^{-1}\frac{\partial}{\partial x}+\frac{\partial}{\partial y}$$

and

$$(2.4)\quad C=x(1-x)\frac{\partial}{\partial x}-y^2\frac{\partial}{\partial y}+(\gamma-1-\beta x)y$$

such that

$$(2.5)\quad B\,f_n(x, y)=\frac{n(\gamma-\beta-n)}{\gamma-n}f_{n-1}(x, y)$$

and

$$(2.6)\quad C\,f_n(x, y)=(\gamma-n-1)\,f_{n+1}(x, y),$$

where $f_n(x, y)=y^n F(-n, \beta ; \gamma-n ; x)$.

3. LIE ALGEBRA :

The set I, A, B, C forms a Lie algebra with the commutator relations

$$[A, B]=-B, \quad [A, C]=C$$

$$(3.1)\quad [B, C]=-2A+\gamma-\beta-1$$

Now

$$CB=(1-x)\;L+(\gamma-\beta-1)\,A,$$

where L denotes the second order partial differential operator used to represent (2.1) in the form $Lu=0$. Thus A, B. C commute with $(1-x)\,L$.

The transformation groups generated by B, C are given by

$$e^{bB} f(x, y) = f\left[\frac{xy+b}{y+b}, \quad y+b\right]$$

(3.2)

$$e^{cC} f(x, y) = (1+cy)^{\gamma-1} (1+cxy)^{-\beta} f\left[x \frac{1+cy}{1+cxy}, \frac{y}{1+cy}\right]$$

Then as $[B. C] \neq 0$, we have

$$(3.3) \quad e^{cC} e^{bB} f(x.y) = (1+cy)^{\gamma-1} (1+cxy)^{-\beta}$$

$$\cdot f\left[\frac{b+(1+bc)xy}{b+(1+bc)y}. \frac{1+cy}{1+cxy}. \frac{b+(1+bc)y}{1+cy}\right]$$

$$(3.4) \quad e^{bB} e^{cC} f(x, y) = (1+bc+cy)^{\gamma-1} (1+bc+cxy)^{-\beta}$$

$$\cdot f\left[\frac{(b+xy)(1+bc+cy)}{(b+y)(1+bc+cxy)}, \frac{y+b}{1+bc+cy}\right]$$

4. GENERATING FUNCTIONS :

PART-I : Genereting functions derived from the first order operator conjugate to A-n. From the previous considerations we know that $u(x, y) = y^n F(-n, \beta; \gamma-n; x)$ is annulled by L and A-n, where $A = y \frac{\partial}{\partial y}$.

To obtain generating functions for $F(-n, \beta; \gamma-n; x)$ we now transform $u(x, y)$ by means of the operators $e^{cC} e^{bB}$ anb $e^{bB} e^{cC}$. We consider ths following cases :

Case — 1 : $b=1. c=0$

Case — 2 : $b=0, c=-1$

Case — 3 : $bc \neq 0$.

Case — 1 : We have

$$e^{B} y^{n} F(-n, \beta; \gamma-n; x)$$

$$= (1+y)^{n} F\left(-n. \beta; \gamma-n; \frac{xy+1}{y+1}\right)$$

On the other hand

$$e^{B} y^{n} F(-n. \beta; \gamma-n; x) = \sum_{k=0}^{\infty} \frac{(-n)_k (\gamma-\beta-n)_k (-1)^k y^{n-k}}{(\gamma-n)_k k!} F(k-n. \beta; \gamma-n+k; x)$$

Equating these two and writing t in place of $-y^{-1}$, we get (1.1)

$$(4.1)\quad (1-t)^n F\left(-n, \beta ; \gamma-n ; \frac{x-t}{1-t}\right)=\sum_{k=0}^{n} \frac{(-n)_k (\gamma-\beta-n)_k}{(\gamma-n)_k\, k!} F(-n+k, \beta ; \gamma-n+k ; x)t^k$$

which is due to Das [1. (6.5)] replacing γ by $\gamma+n$ in (4.1)

Case—2 : We have

$$e^{-C}[y^n F(-n, \beta ; \gamma-n ; x)] = y^n (1-y)^{\gamma-n-1} (1-xy)^{-\beta} F\left(-n, \beta ; \gamma-n ; \frac{1-y}{1-xy} x\right)$$

Also

$$e^{-C}[y^n F(-n, \beta ; \gamma-n ; x)]=\sum_{k=0}^{\infty} \frac{(1-\gamma+n)_k}{k!} y^{n+k} F[-n-k, \beta ; \gamma-n-k ; x].$$

Equating these two we get (1.2)

$$(4.2)\quad (1-y)^{\gamma-n-1} (1-xy)^{-\beta} F\left(-n, \beta ; \gamma-n ; \frac{1-y}{1-xy} x\right) = \sum_{k=0}^{\infty} \frac{(n-\gamma+1)_k}{k!} F(-n-k, \beta ; \gamma-n-k ; x)y^k$$

which is again due to Das [1, (6.6)] replacing γ by $\gamma+n$ in (4.2)

Case—3 : $bc \neq 0$.

We consider $b = w^{-1}$ and $c=-1$. Then from (3.3), we get

$$(4.3)\quad (1-y)^{\gamma-n-1} (1-xy)^{-\beta} (1+(w-1)y)^n F\left[-n, \beta ; \gamma-n ; \frac{(1-y)(1-xy+wxy)}{(1-xy)(1-y+wy)}\right]$$
$$= \sum_{k=0}^{\infty} \frac{y^k}{k!} \sum_{m=0}^{n} \frac{(-n)_m(\gamma-n-\beta)_m(n-\gamma+1-m)_k}{(\gamma-n)_m\, m!\, (-1)^m} (wy)^{n-m} \cdot F(-n-k+m, \beta ; \gamma-n-k+m ; x)$$

and putting $b=-w^{-1}$, $c=-1$. we get from (3.4)

$$(4.4)\quad (wy-1)^n\left(\frac{1+w-wy}{w}\right)^{\gamma-n-1}\left(\frac{1+w-wxy}{w}\right)^{-\beta} F\left[-n, \beta ; \gamma-n ; \frac{(1-wxy)(1+w-wy)}{(1-wy)(1+w-wxy)}\right]$$
$$= \sum_{k=0}^{\infty} \frac{(wy)^{n-k}}{k!} \sum_{m=0}^{k-n} \frac{(1-\gamma+n)_m (-n-m)_k (\gamma-n-m-\beta)_k}{(\gamma-n-m)_k\, m!} \cdot y^m F(-n-m+k, \beta ; \gamma-n-m+k ; x).$$

PART—II : Generating functions derived from operators not conjugate to A—n.

Let $S = e^{cC} e^{bB}$. Then for each choice of b and c, $S(A-n)S^{-1}$ represents an operator conjugate to A—n.

Let $R = r_1A + r_2B + r_3C + r_4$, for all choices of the coefficients except for $r_1 = r_2 = r_3 = 0$. Then $Lu = 0$ and $Ru = 0$ iff $\Psi(x)\ L(Su) = 0$ and $SRS^{-1}(Su) = 0$.

Now we have

$$e^{aA} B e^{-aA} = e^{-a} B,\quad e^{aA} C e^{-aA} = e^{a} C$$

$$e^{bB} A e^{-bB} = A + bB,\quad e^{cC} A e^{-cC} = A - cC$$

$$(4.5)\quad e^{bB} C e^{-bB} = -2bA - b^2B + C + b(\Upsilon-\beta-1)$$

$$e^{cC} B e^{-cC} = 2cA + B - c^2C + c(\beta-Y+l).$$

Then

$$SAS^{-l} = e^{cC}(A+bB)e^{-cC}$$

$$(4,6)\quad = (1+2bc)A + bB - c(1+bc)C + bc(\beta-Y+1).$$

Therefore, for $R = r_1A + r_2B + r_3C + r_4$, A—n is conjugate to R if $r_1^2 + 4r_2r_3 = 1$, so that A—n is not conjugate to the set of operators for which $r_1^2 + 4r_2r_3 \neq 1$.

We choose $r_1^2 + 4r_2r_3 = 0$ for which A—n is not conjugate to $R = r_1A + r_2B + r_3C + r_4$. Then the following cases may arise :

Case—1 : $r_1 = 0,\ r_2 = 1,\ r_3 = 0$

Case—2 : $r_1 = 2,\ r_2 = 1,\ r_3 = -1$

Case—3 : $r_1 = 0\ \ r_2 = 0,\ r_3 = 1$.

Part-II, Case—1 : We find a solution for the system $(B+\eta)u = 0$ and $Lu = 0$, where η is a non—zero constant.

Now

$u = e^{-\eta y} f(y-xy)$, a solution of $(B+\eta)u = 0$, $B = (1-x)y^{-1}\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}$, is again a solution of $Lu = 0$, where

$$L = x(1-x)\frac{\partial^2}{\partial x^2} - (1-x)y\frac{\partial^2}{\partial x\partial y} + (\gamma-(\beta+1)x)\frac{\partial}{\partial x} + \beta y\frac{\partial}{\partial y}.$$

Then f (X), $X = y-xy$, satisfy the following ordinary differential equation

$X f''(X) + (\beta - \gamma + 1 - \eta X) f'(X) - \beta\eta f(X) = 0$

which has a solution (for $\eta = 1$) ${}_1F_1(\beta ; \beta-\gamma+1 ; X)$.

Thus

$u = e^{-y} \; {}_1F_1(\beta ; \beta-\lambda+1 ; y-xy)$ is a solution for the system $Lu = 0$, $(\beta+1) u = 0$. This function can be expanded in powers of y, say in this form,

$$e^{-y} {}_1F_1(\beta ; \beta-\gamma+1 ; y-xy) = \sum_{n=0}^{\infty} a_n \; F(-n, \beta ; \gamma-n ; x) \, y^n$$

Putting $x = 0$ and equating coefficients of y^n from both sides, we get

$$a_n = \frac{(1-\gamma)_n}{(\beta-\gamma+1)_n} \frac{(-1)^n}{n!}.$$

Thus we have

$$(4.7) \quad e^{-y} \, {}_1F_1(\beta ; \beta-\gamma+1 ; y-xy) = \sum_{n=0}^{\infty} \frac{(1-\gamma)_n \, (-y)^n}{(\beta-\gamma+1)_n \, n!} F(-n, \beta ; \gamma-n ; x),$$

which is believed to be new.

Part-II, Case 2 : We are to find a solution for the system

$(2A + B - C + \eta) u = 0$, $Lu = 0$.

We avoid to solve actually, as

$$e^{C} B e^{-C} = 2A + B - C + \beta - \gamma + 1.$$

If u is a solution of $Lu = 0$ and $(B+w) u = 0$, $e^{C} u$ is a solution of $Lu = 0$ and $(2A + B - C + \beta - \gamma + 1 + w) u = 0$.

Now from Part—II, Case—1,

$$u = e^{wy} \; {}_1F_1(\beta ; \beta-\gamma+1 ; -w(1-x) y)$$

is annulled by L and $B-w$.

Then

$$e^{C} u = (1+y)^{\gamma-1} (1+xy)^{-\beta} \exp\left(\frac{wy}{1+y}\right) {}_1F_1\left[\beta ; \beta-\gamma+1 ; \frac{-wy(1-x)}{(1+y)(1+xy)}\right]$$

is annulled by L and $2A + B - C + \beta - \gamma + 1 - w$. This function can be expanded in powers of $-y = t$, say in this from

$$(1-t)^{\gamma-1} (1-xt)^{-\beta} \exp\left(\frac{-wt}{1-t}\right) {}_1F_1\left[\beta ; \beta-\gamma+1 ; \frac{(1-x)wt}{(1-t)(1-xt)}\right]$$

$$= \sum_{n=0}^{\infty} a_n F(-n, \beta ; \gamma-n ; x) \, L_n^{\beta-\gamma}(w) \, t^n.$$

Equating coefficients of t^n from both sides, we get

$$a_n = \frac{(1-\gamma)_n}{(\beta-\gamma+1)_n}.$$

Thus, we obtain a new bilateral generating relation parallel to a result of L. Weisner [2. (4.6)]

(4.8) $$(1-t)^{\gamma-1}(1-xt)^{-\beta} \exp\left(\frac{-wt}{1-t}\right) {}_1F_1\left[\beta ; \beta-\gamma+1 ; \frac{(1-x)wt}{(1-t)(1-xt)}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(1-\gamma)_n}{(\beta-\gamma+1)_n} F(-n, \beta ; \gamma-n ; x) L_n^{\beta-\gamma}(w) t^n.$$

Part-II, Case-3 : We find a solution for the system $Lu = 0$ and $(C+\eta)u = 0$. From Part-II, Case-1,

$$u = e^y {}_1F_1(\beta ; \beta-\gamma+1 ; (x-1)y)$$

is annulled by L and B—1.

As $e^{-B}e^{C}(B-1)e^{-C}e^{B} = -C-1$, we have

$$e^{-B}e^{C}u = x^{-\beta}y^{\gamma-\beta-1} \exp\left(\frac{y-1}{y}\right) {}_1F_1\left(\beta ; \beta-\gamma+1 ; \frac{x-1}{xy}\right)$$

is annulled by L and C + 1.

This function can be expanded in powers of $y^{-1} = t$. Then, we have

(4.9) $$\sum_{n=0}^{\infty} \frac{(1-\gamma)_n}{(\beta-\gamma+1)_n\, n!} F(-n, \beta ; \gamma-n ; x)(-t)^n$$

$$= x^{-\beta} e^{-t} {}_1F_1(\beta ; \beta-\gamma+1 ; (1-x^{-1})t), \quad (x \neq 0),$$

which dos not seem to appear before.

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