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AN EXTENSION OF A THEOREM OF FISHER ON COMMON FIXED POINT

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1. Introduction : In 1977, B. Fisher [2] proved the following theorem :

Theorem 1. In a complete metric space (X, d) if there exist two operators S and T mapping X into itself and satisfying the relation

(1.1) $$[d(Sx, Ty)]^2 \leqslant b\, d(x, Sx)\, d(y, Ty) + c\, d(x, Ty)\, d(y, Sx)$$

for all $x, y \in X$, where $0 \leqslant b < 1$ and $c \geqslant 0$, then S and T have a common fixed point. Further if $0 \leqslant b, c < 1$, then each of S and T has a unique fixed point and these two points coincide.

It may now be pointed out that the metrics d (x, Sx) and d (y, Ty) occur in the contraction mapping of R. Kannan [3] and the metrics d (x, Ty) and d (y, Tx) occur in the contraction mapping of the present author [1]. Thus the product of two metrics appearing in the right member of (1.1) may be varied in six different ways, viz.

$$d(x, Sx)\, d(y, Ty), \quad d(x, Sx)\, d(x, Ty),$$
$$d(x, Sx)\, d(y, Sx), \quad d(y, Ty)\, d(x, Ty),$$
$$d(y, Ty)\, d(y, Sx), \quad d(x, Ty)\, d(y, Sx).$$

In view of the above fact, it may be of interest to remark that Theorem 1 can be extended as follows :

Theorem 2. In a complete metric space (X, d) if there exist two operators S and T mapping X into itself and satisfying the relation

$$[d(Sx, Ty)]^2$$

$$(1.2)\quad \leqslant \alpha_1 d(x, Sx)\, d(y, Ty) + \alpha_2 d(x, Sx)\, d(x, Ty) + \alpha_3 d(x, Sx)\, d(y, Sx) + \alpha_4 d(y, Ty)\, d(x, Ty) + \alpha_5 d(y, Ty)\, d(y, Sx) + \alpha_6 d(x, Ty)\, d(y, Sx)$$

for all $x, y \in X$, where $\alpha_1 \geqslant 0$ and $0 \leq \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$, then S and T have a common fixed point. Further if $\alpha_6 < 1$ then S and T have a unique common fixed point.

2. Proof of Theorem 2.

It follows from (1.2) that

$$(2.1)\quad [d(Ty, Sx)]^2 \leq \alpha_1 d(y, Ty)\, d(x, Sx) + \alpha_2 d(y, Ty)\, d(y, Sx) + \alpha_3 d(y, Ty)\, d(x, Ty) + \alpha_4 d(x, Sx)\, d(y, Sx) + \alpha_5 d(x, Sx)\, d(x, Ty) + \alpha_6 d(y, Sx)\, d(x, Ty).$$

By virtue of the symmetry relation we obtain from (1.2) and (2.1)

$$(2.2)\quad [d(Sx, Ty)]^2 \leq \alpha_1 . d(x, Sx)\, d(y, Ty) + \alpha_6 d(x, Ty)\, d(y, Sx) + \frac{\alpha_2 + \alpha_5}{2}[d(y, Ty)\, d(y, Sx) + d(x, Sx)\, d(x, Ty)] + \frac{\alpha_3 + \alpha_4}{2}[d(y, Ty)\, d(x, Ty) + d(x, Sx)\, d(y, Sx)]$$

Now let $x_0 \in X$ and define

$$x_{2n+1} = Sx_{2n}, \quad n = 0, 1, 2, \ldots$$
$$x_{2n} = Tx_{2n-1}, \quad n = 1, 2, \ldots$$

Thus by (2.2) we have

$$[d(x_{2n+1}, x_{2n})]^2$$
$$= [d(Sx_{2n}, Tx_{2n-1})]^2$$
$$\leqslant \alpha_1 d(x_{2n}, x_{2n+1})\, d(x_{2n-1}, x_{2n})$$
$$+ \frac{\alpha_2 + \alpha_5}{2} d(x_{2n-1}, x_{2n})\, d(x_{2n-1}, x_{2n+1})$$
$$+ \frac{\alpha_3 + \alpha_4}{2} d(x_{2n}, x_{2n+1})\, d(x_{2n-1}, x_{2n+1})$$
$$\leqslant \frac{\alpha_1}{2}[\{d(x_{2n}, x_{2n-1})\}^2 + \{d(x_{2n}, x_{2n+1})\}^2]$$
$$+ \frac{\alpha_2 + \alpha_5}{2} d(x_{2n}, x_{2n-1})[d(x_{2n+1}, x_{2n}) + d(x_{2n}, x_{2n-1})]$$
$$+ \frac{\alpha_3 + \alpha_4}{2} d(x_{2n}, x_{2n-1})[d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n-1})]$$

$$\leqslant \frac{\alpha_1}{2}[\{d(x_{2n}, x_{2n-1})\}^2 + \{d(x_{2n}, x_{2n+1})\}^2]$$

$$+ \frac{\alpha_2+\alpha_5}{2}\Big[d(x_{2n}, x_{2n-1})\Big]^2 + \frac{\alpha_3+\alpha_4}{2}\Big[d(x_{2n}, x_{2n+1})\Big]^2$$

$$+ \frac{\alpha_2+\alpha_5}{4}[\{d(x_{2n}, x_{2n-1})\}^2 + \{d(x_{2n}, x_{2n+1})\}^2]$$

$$+ \frac{\alpha_3+\alpha_4}{4}[\{d(x_{2n}, x_{2n+1})\}^2 + \{d(x_{2n}, x_{2n-1})\}^2]$$

i.e. $$\left(1-\frac{\alpha_1}{2}-\frac{\alpha_2+\alpha_5}{4}-\frac{\alpha_3+\alpha_4}{4}-\frac{\alpha_3+\alpha_4}{2}\right)[d(x_{2n+1}, x_{2n})]^2$$

$$\leqslant\left(\frac{\alpha_1}{2}+\frac{\alpha_2+\alpha_5}{2}+\frac{\alpha_2+\alpha_5}{4}+\frac{\alpha_3+\alpha_4}{4}\right)[d(x_{2n}, x_{2n-1})]^2.$$

Thus

(2.3) $d(x_{2n}, x_{2n+1}) \leqslant k d(x_{2n-1}, x_{2n})$,

$$\text{where } k^2 = \frac{\frac{\alpha_1}{2}+\frac{\alpha_2+\alpha_5}{2}+\frac{\alpha_2+\alpha_5}{4}+\frac{\alpha_3+\alpha_4}{4}}{1-\frac{\alpha_1}{2}-\frac{\alpha_3+\alpha_4}{2}-\frac{\alpha_3+\alpha_4}{4}-\frac{\alpha_2+\alpha_5}{4}} < 1.$$

Similarly we can show

(2.4) $d(x_{2n-1}, x_{2n}) \leqslant k d(x_{2n-2}, x_{2n-1})$.

So $\{x_n\}$ is a Cauchy sequence in X and X being complete, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Now by (1.2) we have

$$[d(Sz, x_{2n})]^2$$
$$= [d(Sz, Tx_{2n-1})]^2$$
$$\leqslant \alpha_1 d(z, Sz)\, d(x_{2n-1}, Tx_{2n-1})$$
$$+ \alpha_2 d(z, Sz)\, d(z, Tx_{2n-1})$$
$$+ \alpha_3 d(z, Sz)\, d(x_{2n-1}, Sz)$$
$$+ \alpha_4 d(x_{2n-1}, Tx_{2n-1})\, d(z, Tx_{2n-1})$$
$$+ \alpha_5 d(x_{2n-1}, Tx_{2n-1})\, d(x_{2n-1}, Sz)$$
$$+ \alpha_6 d(z, Tx_{2n-1})\, d(x_{2n-1}, Sz)$$

Using $n \to \infty$, we thus obtain

$$(1-\alpha_3)[d(Sz, z)]^2 \leqslant 0,$$

which implies $Sz = z$.

Similarly considering $[d(x_{2n+1}, Tz)]^2$ we can show that $Tz = z$

Thns $Sz = z = Tz$.

Finally we prove that z is the unique common fixed point of S and T if $\alpha_6 < 1$.

Let u be any point of X such thet $Su = u$, then we have by (1.2)

$$[d(z, u)]^2 = [d(Su, Tz)]^2$$
$$\leqslant \alpha_6 [d(z, u)]^2,$$

which implies that $z = u$, since $\alpha_6 < 1$,

Similarly we can show that z is the unique fixed point of T.

Lastly if v be any point of X such that $Sv = v = Tv$, then by using (1.2) we can easily prove that $v = z$.

This completes the proof of theorem 2.

References

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