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FIXED POINT THEOREMS IN BANACH SPACES

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In this paper we have proved two fixed point theorems in Banach spaces, which are extensions of the results of K, Goebel and E. Zlotkiewicz [1] and K. Iseki [2]. First we prove the following theorem :—

Theorem 1. Let F be a mapping of a Banach space X into itself satisfying the following conditions :

(a) $F^2 = I$ where I is the identity mapping

(b) $$\| F(x) - F(y) \| \leqslant \frac{a \| x - F(y) \| \quad \| y - F(x) \|}{\| x - y \|} + b\{ \| x - F(x) \| + \| y - F(y) \| \} + c\{ \| x - F(y) \| + \| y - F(x) \| \} + d \| x - y \|$$

for every $x, y \in X$ and $x \neq y$, $0 \leqslant a, b, c, d$, $3a + 4b + 4c + d < 2$.

Then F has a fixed point in X. Further the fixed point is unique if $a + 2c + d < 1$.

Proof. Let x be a point in X. We take $y = \frac{1}{2}(F + I)(x)$, $z = F(y)$ and $u = 2y - z$.

Now $\| z - u \| \leqslant \| z - x \| + \| u - x \|$

Again, $\| z - x \| = \| F(y) - F^2(x) \|$

$$\leqslant \frac{a \| y - F^2(x) \| \quad \| F(x) - F(y) \|}{\| y - F(x) \|}$$

$$+ b\{ \| y - F(y) \| + \| F(x) - F^2(x) \| \}$$

$$+ c\{ \| y - F^2(x) \| + \| F(x) - F(y) \| \} + d \| y - F(x) \|$$

$$= \frac{a \| y - x \| \{ \| F(x) - y \| + \| y - F(y) \| \}}{\| y - F(x) \|}$$

$$+ b\{ \| y - F(y) \| + \| F(x) - x \| \}$$

$$+c\{\|y-x\|+\|F(x)-y\|+\|y-F(y)\|\}$$
$$+d\|\tfrac{1}{2}(F+I)(x)-F(x)\|$$
$$\leqslant a\|\tfrac{1}{2}(F+I)(x)-x\|+\frac{a/2\|x-F(x)\|\ \ \|y-F(y\|}{\|\frac{1}{2}(F+I)x-F(x)\|}$$
$$+b\{\|y-F(y)\|+\|x-F(x)\|\}$$
$$+c\|x-F(x)\|+c\|y-F(y)\|+d/2\|x-F(x)\|$$
$$=\frac{a}{2}\|x-F(x)\|+a\|y-F(y)\|+b\|y-F(y)\|$$
$$+b\|x-F(x)\|+c\|x-F(x)\|+c\|y-F(y)\|$$
$$+\frac{d}{2}\|x-F(x)\|$$
$$=\left(\frac{a}{2}+\frac{d}{2}+c+b\right)\|x-F(x)\|+(a+b+c)\|y-F(y)\|$$

Thus

$$\|u-x\|=\|2y-z-x\|=\|(F+I)(x)-F(y)-x\|$$
$$=\|(F+I)(x)-F(y)-x\|$$
$$=\|F(x)-F(y)\|$$
$$\leqslant\frac{a\|x-F(y)\|\ \ \|y-F(x)\|}{\|x-y\|}+b\{\|x-F x)\|+\|y-F(y)\|$$
$$+c\{\|x-F(y)\|+\|y-F(x)\|\}+d\|x-y\|$$
$$\leqslant\frac{a\{\|x-y\|+\|y-F(y)\|\}\|y-F(x)\|}{\|x-y\|}$$
$$+b\{\|x-F(x)\|+\|y-F(y)\|\}$$
$$+c\{\|x-y\|+\|y-F(y)\|\}+\frac{c}{2}\|x-F(x)\|$$
$$+\frac{d}{2}\|x-F(x)\|$$
$$=\frac{a}{2}\|x-F(x)\|+a\|y-F(y)\|+b\|x-F(x)\|$$
$$+b\|y-F(y)\|+\frac{c}{2}\|x-F(x)\|+c\|y-F(y)\|$$
$$+\frac{c}{2}\|x-F(x)\|+\frac{d}{2}\|x-F(x)\|$$
$$=\left(b+c+\frac{a}{2}+\frac{d}{2}\right)\|x-F(x)\|+(a+b+c)\|y-F(y)\|$$

Also $\|z-u\|=\|F(y)-2y+z\|=2\|F(y)-y\|$

$$\therefore \quad 2\,\|F(y)-y\| \leqslant 2(a+b+c)\,\|y-F(y)\|$$
$$+2\left(\frac{a}{2}+\frac{d}{2}+b+c\right)\|x-F(x)\|$$

Or, $\|y-F(y)\| \leqslant (a+b+c)\,\|y-F(y)\|$
$$+\left(\frac{a}{2}+\frac{d}{2}+b+c\right)\|x-F(x)\|$$

Or, $(1-a-b-c)\,\|y-F(y)\| \leqslant \left(\frac{a}{2}+\frac{d}{2}+b+c\right)\|x-F(x)\|$

Or, $\|y-F(y)\| \leqslant \dfrac{\frac{a}{2}+\frac{d}{2}+b+c}{1-a-b-c}\,\|x-F(x)\|$

$$<\frac{a+d+2b+2c}{2-2a-2b-2c}\,\|x-F(x)\|$$

$$<\alpha\,\|x-F(x)\| \text{ where } \alpha=\frac{a+d+2b+2c}{2-2a-2b-2c}<1$$

since $3a+4b+4c+d<2$

Let

$G=\frac{1}{2}(F+I)$, then for any $x \in X$,

$$\|G^2(x)-G(x)\| = \|G(y)-y\|$$
$$=\left\|\frac{1}{2}(F+I)(y)-y\right\|$$
$$=\frac{1}{2}\,\|y-F(y)\| < \frac{\alpha}{2}\,\|x-F(x)\|$$

$\therefore$ $\{G^n(x)\}$ is a Cauchy sequence in X.

As X is complete, $\{G^n(x)\}$ converges to some element x_0 in X, i.e $\lim\limits_{n\to\infty} G^n(x)=x_0$

i.e. $G(x_0)=x_0$

Hence $F(x_0)=x_0$.

i.e. x_0 is a fixed point of X.

Now to show that the fixed point is unique, let us suppose that if possible y_0 be another fixed point of X. Then

$$\|x_0-y_0\| = \|F(x_0)-F(y_0)\|$$

$$\leqslant \frac{a\,\|x_0-F(y_0)\|\,\|y_0-F(x_0)\|}{\|x_0-y_0\|}+b\{\|x_0-F(x_0)\|+\|y_0-F(y_0)\|\}.$$

$$+c\{\|y_0-F(x_0)\|+\|x_0-F(y_0)\|\}+d\,\|x_0-y_0\|$$

i.e. $(1-a-2c-d)\,\|x_0-y_0\| \leqslant 0 \Rightarrow x_0=y_0$ since $a+2c+d<1$.

Corollaries :

i) If we put $a=0$, $b=0$, $c=o$, we get result of K. Goebel and E. zlotkiewicz. [i]

ii) If we put $a=0$, $c=0$, we get result of K. Iseki. [2]

Next we prove the following theorem :

Theor em 2: Let F_1 and F_2 be two non expansive mappings of a Banach space X into itself and F_1 and F_2 satisfy the following conditions

i) $F_1 F_2=I=F_2 F_1$ where I is the identity mapping.

and

ii)
$$\| F_1(x)-F_2(y) \| \leqslant \frac{a \| x-F_2(y) \| \, \| y-F_1(x) \|}{\| x-y \|}$$
$$+b\{ \| x-F_1(x) \| + \| y-F_2(y) \| \}$$
$$+c\{ \| x-F_2(y) \| + \| y-F_1(x) \| \}$$
$$+d \| x-y \|$$

for every $x \neq y \in X$, $0 \leqslant a, b, c, d$, and $3a+4b+4c+d<2$.

Then F_1 and F_2 have a common fixed pojnt. Further the fixed point is unique if

$$a+2c+d<1.$$

Proof. Let $y=\frac{1}{2}(F_2+I)(x)$, $z=F_1(y)$

$$u=2y-z.$$

$$\| z-x \| = \| F_1(y)-F_1 F_2(x) \|$$
$$\leqslant \frac{a \| y-F_1 F_2(x) \| \, \| F_2(x)-F_1(y) \|}{\| y-F_2(x) \|}$$
$$+b\{ \| y-F_1(y) \| + \| F_2(x)-F_1 F_2(x) \| \}$$
$$+c\{ \| y-F_1 F_2(x) \| + \| F_2(x)-F_1(y) \| \}$$
$$+d \| y-F_2(x) \|$$
$$= \frac{a \| \frac{1}{2}(F_2+I)(x)-F_1 F_2(x) \| \, \| F_2(x)-F_1(y) \|}{\| \frac{1}{2}(F_2+I)(x)-F_2(x) \|}$$
$$+b\{ \| \tfrac{1}{2}(F_2+I)(x)-F_1(y) \| + \| F_2(x)-F_1 F_2(x) \| \}$$
$$+c \| \tfrac{1}{2}(F_2+I)(x)-F_1 F_2(x) \| + \| F_2(x)-F_1(y) \|$$
$$+d \| \tfrac{1}{2}(F_2+I)(x)-F_2(x) \|$$

$$= \frac{\frac{a}{2} \| F_2 x)-x \| \, \| F_2(x)-F_1(y) \|}{\frac{1}{2} \| F_2(x)-x \|}$$

$$+b\{\|y-F_1(y)\|+\|F_2(x)-x\|\}$$

$$+c\{\|\tfrac{1}{2}(F_2+I)(x)-x\|+c\|F_2(x)-F_1(y)\|$$

$$+\frac{d}{2}\|x-F_2(x)\|$$

$$\leqslant\frac{a\|x-F_2(x)\|\{\|F_2(x)-y\|+\|y-F_1(y)\|\}}{\|x-F_2(x)\|}$$

$$+b\|y-F_1(y)\|+b\|x-F_2(x)\|+\frac{c}{2}\|x-F_2(x)\|$$

$$+\frac{c}{2}\|x-F_2(x)\|+c\|y-F_1(y)\|+\frac{d}{2}\|x-F_2(x)\|$$

$$=\left(\frac{a}{2}+b+c+\frac{d}{2}\right)\|x-F_2(x)\|+(a+b+c)\|y-F_1(y)\|$$

Thus

$$\|u-x\|=\|2y-z-x\|$$

$$=\|(F_2+I)(x)-F_1(y)-x\|$$

$$=\|F_2(x)-F_1(y)\|$$

$$\leqslant\frac{a\|x-F_1(y)\|\ \|y-F_2(x)\|}{\|x-y\|}$$

$$+b\{\|y-F_1(y)\|+\|x-F_2(x)\|\}$$

$$+c\{\|y-F_2(x)\|+\|x-F_1(y)\|\}$$

$$+d\|x-y\|$$

$$\leqslant\frac{a\{\|x-y\|+\|y-F_1(y)\|\}\frac{1}{2}\|x-F_2(x)\|}{\frac{1}{2}\|x-F_2(x)\|}$$

$$+b\{\|y-F_1(y)\|+\|x-F_2(x)\|\}$$

$$+\frac{c}{2}\|x-F_2(x)\|+c\|x-y\|$$

$$+c\|y-F_1(y)\|+\frac{d}{2}\|x-F_2(x)\|$$

$$=\frac{a}{2}\|x-F_2(x)\|+a\|y-F_1(y)\|+b\|y-F_1(y)\|$$

$$+b\|x-F_2(x)\|+\frac{c}{2}\|x-F_2(x)\|+\frac{c}{2}\|x-F_2(x)\|$$

$$+c\|y-F_1(y)\|+\frac{d}{2}\|x-F_2(x)\|$$

$$=\left(\frac{a}{2}+\frac{d}{2}+b+c\right)\|x-F_2(x)\|+(a+b+c)\|y-F_1(y)\|$$

$$\therefore \quad \|z-u\| \leqslant \|z-x\|+\|u-x\|$$

$$\leqslant 2\left(\frac{a}{2}+\frac{d}{2}+b+c\right)\|x-F_2(x)\|$$

$$+2(a+b+c)\|y-F_1(y)\| \quad \ldots\ldots\ldots \quad (1)$$

Also,

$$\|z-u\| = \|F_1(y)-2y+z\|$$

$$= \|F_1(y)-2y+F_1(y)\|$$

$$=2\|y-F_1(y)\| \quad \ldots\ldots\ldots \quad (2)$$

From (1) and (2) we have

$$2\|y-F_1(y)\| \leqslant 2\left(\frac{a}{2}+\frac{d}{2}+b+c\right)\|x-F_2(x)\|$$

$$+2(a+b+c)\|y-F_1(y)\|$$

$$\therefore \quad \|y-F_1(y)\| \leqslant \left(\frac{a}{2}+\frac{d}{2}+b+c\right)\|x-F_2(x)\|$$

$$+(a+b+c)\|y-F_1(y)\|$$

$$\text{Or,} \quad \|y-F_1(y)\| \leqslant \frac{2b+2c+a+d}{2-2(a+b+c)}\|x-F_2(x)\|$$

$$=\alpha\|x-F_2(x)\| \quad \text{where}$$

$$\alpha=\frac{2b+2c+a+d}{2-2(a+b+c)}<1 \text{ since } 3a+4b+4c+d<2$$

Let $G=\frac{1}{2}(F_2+I)$, then for any $x \epsilon X$,

$$\|G^2(x)-G(x)\| \quad \|G(y)-y\|$$

$$= \left\|\frac{1}{2}(F_2+I)y-y\right\|$$

$$=\frac{1}{2}\|y-F_2(y)\|$$

$$=\frac{1}{2}\|F_2F_1(y)-F_2(y)\|$$

$$\leqslant \frac{1}{2}\|F_1(y)-y\|, \text{ since } F_1 \text{ is nonexpansive.}$$

$$\leqslant \frac{\alpha}{2}\|x-F_2(x)\|$$

$\therefore$ $\{G^n(x)\}$ is a Cauchy sequence in X. Also by the completeness, $\{G^n(x)\}$ converges to some element x_0 in X,

i. e. $\lim_{n\to\alpha} G^n(x)=x_0$ which implies

$$G(x_0)=x_0$$

Hence $F_2(x_0)=x_0$ i.e. x_0 is a fixed point of F_2.

Again,

$$\| G^2(x)-G(x) \| \leq \frac{\alpha}{2} \| x-F_2(x) \|$$

$$=\frac{\alpha}{2} \| F_2F_1(x)-F_2(x) \|$$

$$\leqslant\frac{\alpha}{2} \| x-F_1(x) \| \quad \text{since } F_2 \text{ is non-expansive.}$$

Therefore we can conclude that

$F_1(x_0)=x_0$ i.e. x_0 is a fixed point of F_1.

Hence $F_1(x_0)=x_0=F_2(x_0)$. Therefore x_0 is a common fixed point of F_1 and F_2.

Proof of the uniqueness of the common fixed point is omitted for the sake of brevity.

REFERENCES

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