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EMBEDDING OF A REGULAR RING IN A REGULAR RING WITH IDENTITY.

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1. Regular rings were first introduced by J. V. Neumann. In his definition of regular rings, the presence of identity was always assumed and the whole theory was developed with that assumption Subsequently many prominent mathematicians avoided this assumption of identity. So the question naturally arises under what condition a regular ring can be imbedded in a regular ring with identity In this paper, an attempt has been made to attack this problem.

2. By a regular ring R we mean an associative ring in which $axa=a$ is solvable for all a in R.

Proposition 1. For any regular ring R, aR is a principal right ideal generated by $a\epsilon R$ and $aR=eR$, where e is an idempotent in R.

Proof : Let $a\epsilon R$, then $\exists x\epsilon R/axa=a$. Put $xa=f$, then $f^2=f$ and $af=a$. Thus $a=af\epsilon aR$.

Thus aR is a principal right ideal generated by a.

Let $z\epsilon aR$, Put $ax=e$, then $ea=a$, $e^2=e$.

Thus $z\epsilon aR \Rightarrow z=ar$, for some $r\epsilon R$.

$\Rightarrow z=ear\epsilon eR$.

Thus $aR \subseteq eR$. ....(1)

Now for any $p \epsilon R$,

$ep\epsilon eR \Rightarrow ep=axp=aq\epsilon aR$.

$eR \subseteq aR$ ......(2)

Hence $aR=eR$ by (1) and (2).

Proposition 2. For any two principal right ideals eR and fR of a regular ring R, we have $eR+fR=(e+g)R$, where g is an idempotent with $gR=(f-ef)R$.

Proof. As $g=g.g=(f-ef)x$, then $eg=0$, which shows $e=(e+g)-(e+g)g$. Put $e+g=a$, then $a\epsilon R \Rightarrow \exists x\epsilon R$ such that $axa=a$. Let $xa=k$, then $k^2=k$ and $ak=a$.

Thus $e=ak-ag=a(k-g)=(e+g)(k-g)\epsilon(e+g)R$.

But $f-ef=g(f-ef)$ as by proposition 1, $(f-ef)\epsilon gR$.

Therefore $f=ef+g(f-ef)+e(f-ef)$.

$=ef+(e+g)(f-ef)\epsilon(e+g)R$.

Hence $eR+fR\subseteq(e+g)R$. ....(1)

On the othe hand $e+g=e+(f-ef)\alpha=e(e-f\alpha)+f\alpha\epsilon eR+fR$.

and $(e+g)R\subseteq eR+fR$ ....(2)

From (1) and (2) we get $(e+g)R=eR+fR$.

Proposition 3. For any two principal right ideals eR and fR of a regular ring R, we have $eR\cap fR=(f-fg)R$, where g is an idempotent with $Rg=R(f-ef)$.

Proof : Indeed $f-fg=f(f-g)\epsilon fR$

$f-fg=(f-ef)+(ef-fg)=(f-ef)g+(ef-fg)$.

[as by proposition 1, $(f-ef)\epsilon Rg$.]

$=e(f-fg)\epsilon eR$.

Then $(f-fg)R\epsilon fR\cap eR$. ....(1)

On the otherhand, let $x\epsilon eR\cap fR$.

Then $x=ex=fx$ and $g=\beta(f-ef)$.

Thus $x=x-f\beta(x-x)=x-f\beta(f-ef)x$

$=x-fgx=(f-fg)x$. Hence $eR\cap fR\subseteq(f-fg)R$ ......(2)

From (1) and (2) we have $eR\cap fR=(f-fg)R$.

Proposition 4. Let e be a given idempotent in a regular ring R. Then the set of all idempotents $f\epsilon R$ such that $eR=fR$ is exactly the set : $\{e+(ey-eye)\ ;\ y\epsilon R\}$.

Proof : First we prove that $eR=fR$ if and only if $e=fe$, $f=ef$. Now these two equations themselves imply that f is idempotent, $f^2=f.f.=f(ef)=(fe)f=ef=f$.

Hence the equations alone characterize the elements f.

Let us define x by the relation $f=e+x$. Then the relation $e=fe$, $f=ef$ means $e=(e+x)e$; $e+x=e(e+x)$ i.e. $xe=0$, $ex=x$. The latter two equations clearly hold if x is of the form $x=ey-eye$ and conversely imply $x=ey-eye$ with $y=x$. Hence our elements f are given by $f=e+(ey-eye)$, $\forall y\epsilon R$.

Proposition 5. If R is a regular ring, then an idempotent $e\epsilon R$ is central if and only if $ey-eye=0$ for every $y\epsilon R$.

Proof : If e is central, then $ey-eye=ey-ey=0$, $\forall\ y\epsilon R$ Conversely let $ey-eye=0\ \forall y\epsilon R$ ....(1)

Now $ye - eye \in R$ Thus by regularity of R, $\exists a \in R$

such that $(ye - eye)\, a.\, (ye - eye) = (ye - eye)$.

Hence $ye - eye = (yea - eyea)(ye - eye)$

$= (yeaye - eyeaye - yeaeye + eyeaeye)$.

$= y(ea - eae)\, ye - ey(ea - eae)ye$

$= 0$, (from 1).

Thus $ye = eye = ey$, $\forall y \in R$ Thus e is central.

From propositions (4) and (5) we get a principal right ideal eR is uniquely generated iff e is central.

Thus by propositions (1), (2) & (3), for a regular ring R, the set {eR} of all principal right ideals forms a lattice $\{\mathcal{R}(R), \cup, \cap\}$ with respect to usual set inclusion relation.

This lattice need not be complete. If for a collection of elements $\{e_\lambda R\}$ of $\mathcal{R}(R)$ we can find a unique element e, for which eR is the lub of $\{e_\lambda R\}$ and in that case, by propositions (4) and (5) e is central in R, we define lub $\{e_\lambda\} = e$.

Proposition 6 If C (R) is the centre of a regular ring R, then C (R) is also a regular ring,

Proof : Let $a \in C(R)$ Then $a \in R \Rightarrow axa = a$ for some $x \in R$.

Define $y = ax^2$ then $aya = a.ax^2a = (axa)xa = axa = a$.

Also $u \in R \Rightarrow yu = ax^2u = x^2ua = x^2uaxa = x^2a^2ux$.

$= xu(axa)x = xa^2ux^2 = aux^2 = uax^2 = uy \Rightarrow y \in C(R)$

Thus C (R) is a regular ring.

Definition : By a complete direct sum $\overset{C}{\Sigma}\oplus R_\alpha$ of the rings R_α we mean the set of all infinite rows $\{r_\alpha\}$ where $r_\alpha \in R_\alpha$

If we define equality, addition and multiplication componentwise, then $\overset{C}{\Sigma}\oplus R_\alpha$ is a ring.

Proposition 7. If $\{R_\alpha\}$ be any collection of regular rings, then the complete direct sum $\overset{C}{\Sigma}\oplus R_\alpha$ of all regular rings R_α is also a regular ring.

Proof : Let $\{r_\alpha\}$ be any element of $\overset{C}{\Sigma\oplus} R_\alpha$ where $r_\alpha \in R_\alpha$ then each R_α being regular,

$\exists x_\alpha \in R_\alpha$ such that $r_\alpha x_\alpha r_\alpha = r_\alpha$

Let us collect all $x_\alpha \in R_\alpha$ and form the infinite row $\{x_\alpha\}$.

Then $\{x_\alpha\} \in \overset{C}{\Sigma\oplus} R_\alpha$ and $\{r_\alpha\}\{x_\alpha\}\{r_\alpha\} = \{r_\alpha\}$.

Hence $\overset{C}{\Sigma\oplus} R_\alpha$ is a regular ring.

3. We now proceed to prove our main theorem viz, the theorem of embeddability of a regular ring into a regular ring with identity.

Theorem 1. Let R be a regular ring with C (R) its centre such that annihilator (C(R))=(0) in R. Then R can be embedded in a regular ring R′ with 1, where 1=lub $\{e_\lambda\}$ and $\{e_\lambda\}$ denoting the set of all central idempotents in R.

Proof : Let R be any regular ring, C(R) is its centre.

Let $e \in C(R)$, then eR is a regular ring with e as the identity.

Let $R' = \overset{C}{\Sigma\oplus}(e_\nu R)$, $e_\nu \in C(R)$

By proposition 7, R′ is a regular ring with identity

Now (......, e, ..., f, ... , g...) is the identity of R,

where {...e, ...f, .. g, ...} denotes the total collection of central idempotents of R.

We define ϕ ; $R \to R'$ as follows :

$a\phi = (\ldots\ldots, ea, \ldots\ldots, fa, \ldots\ldots ga, \ldots\ldots)$

Then $(a+b)\phi = (\ldots\ldots, e(a+b), \ldots\ldots f(a+b), \ldots\ldots, g(a+b), \ldots\ldots)$

$= (\ldots\ldots, ea, \ldots\ldots, fa, \ldots\ldots, ga, \ldots\ldots) + (\ldots\ldots, eb, \ldots\ldots, fb, \ldots\ldots, gb, \ldots\ldots)$

$= a\phi + b\phi$

$(ab)\phi = (\ldots\ldots, e(ab), \ldots\ldots, f(ab), \ldots\ldots, g(ab), \ldots\ldots)$

$= (\ldots\ldots, (ea)(eb), \ldots\ldots, (fa)(fb), \ldots\ldots, (ga)(gb), \ldots\ldots)$

$= (\ldots\ldots, ea, \ldots\ldots, fa, \ldots\ldots, ga, \ldots\ldots)(\ldots\ldots, eb, \ldots\ldots, fb, \ldots\ldots, gb, \ldots\ldots)$

$= (a\phi)(b\phi)$

Now to prove injectivity,

let $a\phi = 0$. Now $a \in R \Rightarrow \exists x \in R / axa = a$.

Put $h = xa$ then $h^2 = h$ and $h\phi = (x\phi)(a\phi) = 0$.

Therefore (... .., he, ..., hf, ..., hg,......)=0

[e, f, g.........∈C(R), he=eh etc]

$\Rightarrow he_\nu = 0, \ \forall e_\nu \in C(R)$

We have to show h=0

Let p∈C(R), ∃ idempotent l∈C(R)/pl=p.

Now hl=0⇒hp=0,

thus h ∈ Ann (C (R))⇒h=0 by hypothesis.

Thus R is embedded in R′.

Therefore from now on an element of R can be considered as an element of R′.

Thus e∈C(R) will take the form e = (..., e, ..., fe, ..., ge, ...) in R′

And eR′=(..., e, .. , fe, ..., ge, ...)R′

Let $\{ e_\nu \}$ be the set of all idempotents in C (R).

Let $\cup \{ e_\nu R' \} = I$ then I is a both sided ideal in R′

We will show I=R′

To show this we will use induction to prove that :

(.., e, ..., f, ..., g, ..) ∈ I

Let $\mathcal{A}$ denote the set of all possible non-null rows obtained from the identity [i. e. the row (..., e, , f, ..., g, ...)] replacing some or none components by zero.

Note that $\mathcal{A}$ is a subset of R′.

Also (, e, ..., f, ..., g,)∈ $\mathcal{A}$

Let A, B∈ $\mathcal{A}$

We define A⩽B, iff AB=A.

Then A⩽A, as AA=A.

Let A⩽B, B⩽A, then AB=A and BA=B.

But AB=BA, therefore A=B.

Let A⩽B, B⩽C, then AB=A, BC=B,

thus AC=(AB) C=A (BC)=AB=A,

Hence⩽is a relation of partial order.

Note that the minimal elements of $\mathcal{A}$ are of the form (..., 0, ..., 0, ..., e, ...) etc

i. e. the rows obtained from (..., e, ..., f, ..., g, ...)

by replacing all but only one component by zero.

We define a property P on A as follows :

$A \in \mathcal{A}$ is said to have P iff $A \in I$

Now the minimals of $\mathcal{A}$ have property P.

Indeed $(\ldots, e \ldots, ef, \ldots, eg, \ldots) \in I$

and $(\ldots, 0, e, \ldots, 0, \ldots, 0, \ldots) \in R'$

I being both sided ideal, the product of these two elements viz.

$(\ldots, 0, \ldots, e, \ldots, 0, \ldots) \in I$

Suppose the property P holds for every $X \leqslant A$

We will show that property holds for A. Let B be any element of $\mathcal{A}$ but $B \underset{\neq}{<} A$,

then $B \in I$ by hypothesis.

Put $C = A - B [A, B \in R' \Rightarrow A - B$ is defined]

Evidently $C \in A$,

thus $CA = (A - B)A = A - BA = A - B = C$,

thus $C \leqslant A$. But $C \neq A$. For otherwise $C = A \Rightarrow B = 0$, contradiction as B is a non-null row.

Hence $C \in I$. That is, $A = B + C \in I$.

Thus by induction hypothesis every element of $\mathcal{A}$ has property P.

So $(\ldots, e, \ldots f, \ldots, g, \ldots) \in I$

Hence $I = R'$

If possible $\cup \{ e_\nu R' \} = gR'$

$$e_\nu \in C(R)$$

Now $R' = gR' \Rightarrow 1 = g.x \Rightarrow g = g.x = 1$

Hence lub $\{ e_\nu \} = 1$

This completes the proof.

4. Let us now study the situation, when a regular ring is embedded in a regular ring with identity. To do this, we start with the following proposition.

Proposition 8. Let C denote the set of all central idempotents in a regular ring R, then $C^*. C^* = C^*$

Proof : Let $ef \in C^*. C^*$, then $(ef)x = ef\,(x) = e\,(xf)$

$= x(ef) \Rightarrow ef \in$ cent R.

Also $(ef)^2=(ef)(ef)=e^2f^2=ef$

Thus $ef\epsilon C^*$. Conversely let $e\epsilon C^*$, then $e=e.e\epsilon C^*.C^*\Rightarrow C^*.C^*=C^*$

Theorem 2. If a regular ring R is imbedded in a regular ring R′ with identity, then we can construct a regular ring D with identity having the properties.

i) C (R) can be imbedded in D.

ii) Annihilator (C (R)) = (0) in D.

Proof : Suppose R is imbedded in R′ with identity and C (R) denote its centre. We define a bina y relation on R′ as follows $a \sim b$ iff $ae_\lambda = be_\lambda, \forall e_\lambda \epsilon C(R)$

Then i) $a \sim a$

ii) $a \sim b \Rightarrow b \sim a$

ii) Let $a \sim b$, $b \sim c$ hold

Then $ae_\lambda = be_\lambda, \forall e_\lambda \epsilon C(R)$

$be_\nu = ce_\nu, \forall e_\nu \epsilon C(R)$

Thus $ae_\lambda e_\nu = ce_\lambda e_\nu, \forall e_\lambda, e_\nu \epsilon C(R)$

Now by proposition 8, $a e_p = ce_p, \forall e_p \epsilon C(R)$

Thus $a \sim c$.

Hence $\sim$ defines an equivalence relation on R′ and partitions elements of R′ into disjoint classes.

Let (a) denote the class corresponding to a.

Let $D=\{(a)\}$

We define in D, +, as follows.

$(a)+(b)=(a+b)$

$(a)(b)=(ab)$

Operations are well defined.

Indeed $(b)=(c)\Rightarrow be_\lambda = ce_\lambda, \forall e_\lambda \epsilon C(R)$

Also $(a+b)e_\lambda = ae_\lambda + be_\lambda = ae_\lambda + ce_\lambda = (a+c)e_\lambda, \forall e_\lambda \epsilon C(R)$

$\Rightarrow (a+b)=(a+c)\Rightarrow (a)+(b)=(a)+(c)$

Similarly $(ab)e_\lambda = (ae_\lambda)(be_\lambda) = (ae_\lambda)(ce_\lambda) = (ac)e_\lambda \; \forall e_\lambda \epsilon C(R)$

$\Rightarrow (a)(b)=(a)(c)$.

Let $(a) \in D$

Then $a \in R$, and by regularity, $\exists x \in R / axa = a$.

Therefore $(axa) = (a)$. Hence $(a)(x)(a) = (a)$.

Thus D is regular.

(1) is the identity of D. Note that (1) contains those elements $x \in R'$ such that $xe_\lambda = 1e_\lambda = e_\lambda$, $\forall e_\lambda \in C(R)$

Thus D is a regular ring with identity.

i) Let $a \in C(R)$.

We define ϕ : $C(R) \to D$

by $a\phi = (a)$, $\forall a \in C(R)$

Now $(a+b)\phi = (a+b) = (a) + (b) = a\phi + b\phi$

$(ab)\phi = (ab) = (a)(b) = (a\phi)(b\phi)$

Let $a\phi = (0)$, so $(a) = (0)$, $ae_\lambda = 0$, $\forall e_\lambda \in C(R)$

Now C(R) being regular, $\exists e \in C(R) / ae = a$,

thus $aee_\lambda = 0$, $\forall e_\lambda \in C(R)$

In particular $ae = 0$, as $e \in C(R)$.

$\Rightarrow a = 0$, thus ϕ is injective and C(R) in imbedded in D.

ii) Let $(g) \in \text{Ann}(C(R))$

$\Rightarrow (g)(e_\lambda) = (0)$, $\forall e_\lambda \in C(R)$.

$\Rightarrow (ge_\lambda) = (0) \Rightarrow ge_\lambda e_\nu = 0$, $\forall e_\lambda . e_\nu \in C(R)$

$\Rightarrow ge_p = 0$, $\forall e_p \in C(R)$, by propositions.

$\Rightarrow (g) = (0)$

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REFERENCES

1. J. V. Neumann. Continuous Geometry University Press 1960.
2. M. C Waddel. Properties of regular ring—Duke Math Journal J 19 (1952) 623-627.
3. A. G. Kurosch. Lectures in General Algebra. London Pergamon Press 1965.

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