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ON SEMILINEAR TENSOR PRODUCT

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ABSTRACT : The object of this paper is to introduce the concept of semilinear tensor product of normed linear spaces over fields with real valued valuations and to consider norm on such tensor product spaces.

1.1. INTRODUCTION :

F.F. Bonsall and J. Duncun [2] have introduced norms in tensor product of normed linear spaces over the field of real numbers. We have, in this paper, introduced semilinear tensor product of normed linear spaces over fields having real valued valuations, using semilinear transformation [4] and finally we have discussed norm (weak) on such tensor product spaces.

1.2. Normed linear space over a field with real valuation.

DEFINITION 1.2.1. :

A linear space X over a field F, with a real valued valuation p is said to be a normed linear space over F, if there exists a map $X \to R$ (denoted by $\| \ \|$) s. t.

i) $\| x \| \geqslant 0$ $\quad \| x \| = 0$ iff $x=0$, $x \in X$.

ii) $\| x+y \| \leqslant \| x \| + \| y \|$, $y \in X$

iii) $\| \alpha x \| = p(\alpha) \| x \|$, $(\alpha \in F)$

Remarks : 1) Under the above definition F is a normed linear space over itself where $\| \alpha \| = p(\alpha)$.

2) It is known that [1], if K be a field with real valued valuation, then the Cauchy sequences over K form a commutative ring Λ containing the identity element, and the null sequences form a maximal ideal N of Λ. Hence the quotient ring Λ/N is a field. Now let $\{a\}$ be the sequence, every element of which is $a \in K$, then $\{a\}$ is obviously a Cauchy sequence, it is called a constant sequence, the corresponding class $(\{a\})$ is called the

principal class. It can be shown that the principal classes belonging to Λ/N. form a subfield isomorphic to k. Hence there exists an extension Ω of K isomorphic to Λ/N. Ω is called the derived field of K. Therefore, *identifying the elements of K with the corresponding principal classes in* Λ/N, K may be imbedded in Λ/N and Ω may then be considered to be identical with Λ/N. It can be verified that the derived field Ω of K, with real valued valuation ϕ, is a completion of K in the sense that

i) Ω has a real valuation Ψ which is an extension of ϕ, where $\Psi(A) = \underset{n\to\infty}{\text{Lt.}}\ \phi(a_n)$,

for any $\{a_j\} \in A$,

A denoting the residue class in Λ/N.

ii) Ω is complete w.r. to Ψ.

iii) K is dense in Ω.

Following remark (2), we have the following definition.

1.2.2. A map $f : X \to F$, X belng an arbitrary set and F being an arbitrary field with a real valued valuation p is said to be bounded on X, if $\| [\{f(x)\}] \| \leqslant M$, M being an arbitrary real number and $[\{f(x)\}] \in \Omega$, Ω being the derived field of F.

i. e. if $p' [\{f(x)\}] \leqslant M$, $\{f(x)\}$ being a Cauchy Sequence over the set $(f(x)) \subset F$, $[\{f(x)\}] \in \Lambda/N$, where Λ is the set of all Cauchy Sequences over $(f(x))$, N is the set of all null (Cauchy) sequences over $(f(x))$ and p′ is the extension of the real valued valuation p of F.

1.2.3. A linear map $f : X \to F$, X being a normed linear space over an arbitrary field of scalars F with a real valued valuation p is said to be bounded on X, if $\| [\{f(x)\}] \| \leqslant M \| x \|$, $\forall\ x \in X$, M being real.

Since in the definition of Ω, each element $f(x) \in F$ has been identified with the corresponding principal class of $f(x)$, norm of $f(x)$ will be given by $\| f(x) \| = p\ f(x)$, where p is the restriction of p′ to the set $(f(x))$. Hence it follows from the above definition that f is continuous on X iff f is bounded on X.

From now on we shall mean by X(F), a normed linear space over an arbitrary field of scalars F, with a real valued valuation p.

PROPOSITION 2. 1.

The set BL (X, F) of all bounded (continuous) linear maps from X(F) of F is a Banach space under pointwise addition and scalar multiplication and norm defined by

$$\| f \| = \sup \| [\{f(x)\}] \|, \quad \| x \| \leqslant 1.$$

$$f \in BL\ (X, F). \quad [\{f(x)\}] \in \Omega.$$

PROOF : BL (X, F) is obviously a linear space over F under pointwise addition and scalar multiplication. To show that it is a normed linear space, we note that.

i) $\| f \| \geq 0$, $\| f \| = 0$ iff $f = 0$ [By definition]

ii) $\| f+g \| \leq \| f \| + \| g \|$ [By definition]

iii) $\| \alpha f \| = p(\alpha) \| f \|$, for

$$\| \alpha f \| = \sup_{\| x \| \leqslant 1} p' [\{(\alpha f)\, x\}] = \sup_{\| x \| \leqslant 1} p' [\{\alpha (f(x)\}] \text{ by definition}$$

$$= \sup_{\| x \| \leqslant 1} p'[\{\alpha\} \{f(x)\}], \{\alpha\} \text{ being the constant Cauchy sequence } \{\alpha, \alpha, \ldots\ldots, \alpha\}.$$

$$= \sup_{\| x \| \leqslant 1} p' [\{\alpha\}] \sup_{\| x \| \leqslant 1} p' [\{f(x)\}]$$

$$= p(\alpha) \| f \|$$

Hence BL (X, F) is a normed linear space over F.

To show that BL (X, F) is complete.

Let {fn} be a Cauchy Sequence in BL (X, F), then

$\| f_m - f_n \| < \epsilon$, for m, n $\geqslant$ N (Intger). Hence for some fixed

$$x \in X, \sup_{\| x \| \leqslant 1} \| [\{(f_m - f_n)\, x\}] \| = \sup_{\| x \| \leqslant 1} \| [\{f_m (x) - f_n (x)\}] \|$$

$$= \sup_{\| x \| \leqslant 1} \| [\{f_m (x)\}] - [\{f_n (x)] \| < \epsilon, \text{ for } m,\ n \geqslant N$$

Thus the classes $[\{f_n(x)\}]$ form Cauchy sequences in Ω, therefore, Ω being complete, the Cauchy sequence of $[\{f_n (x)\}]$ converges in Ω.

Hence $[\{f_n (x)\}]$ tends to $[\{f(x)\}] \in \Omega$, as $n \to \infty$.

Since $x \in X$ is arbitrary, this defines a mapping $f : X \to F$. It remains to show that f is linear, f is bounded and that $f_n \to f$ as $n \to \infty$.

But these are routine verifications by considering Ω as a complete normed linear space. Therefore, BL (X, F) is a Banach space over F.

PROPOSITION : 2.2. We may similarly prove that the set B (X, F) of all bounded maps $f : X \to F$, X being an arbitrary set and F being an arbitrary field with a real valued valuation is a Banach space over F under pointwise addition and scalar multiplication and norm defined by $\| f \| = \sup_{x \in X} \| [\{f(x)\}] \|$

DEFINITION : 2.4. Let X (F_1) and Y (F_2) be normed linear spaces over arbitrary fields F_1 and F_2, which are respectively isomorphic to an arbitrary field F, with a real valuation, under isomorphisms ξ_1 and ξ_2 ; then a map $f : X \times Y \to F$ is called a bounded bi-semilinear map, if.

$\| [\{f(x, y)\}] \| \leq \quad \| M \| \quad \| x \| \quad \| y \|,$

$x \epsilon X$, $y \epsilon y$, $[\{f(x, y)\}] \epsilon \Omega$, Ω being derived field of F.

[A mapping $\Psi : X \times Y \to F$ is called bi-semilinear if

$\Psi \; r_1 x_1 + s_1 x_2, y) = \xi_1 r_1 \; \Psi \; (x_1, y) + \xi_1 s_1 \; \Psi \; (x_2, y)$

$\Psi \; (x, r_2 y_1 + s_2 y_2) = \xi_2 r_2 \; \Psi \; (x, y_1) + \xi_2 s_2 \; \Psi \; (x, y_2)$:

$x, x_1, x_2 \epsilon X$; $y, y_1, y_2 \epsilon y$; $s_1, r_1 \epsilon F_1$; $r_2, s_2 \epsilon F_2$.

Now proceeding as in Prop. 2. 1, we may prove

PROPOSITION : 2.3. The set BL (X, Y ; F) of all bounded bi-semilinear maps $f : X \times Y \to F$ is a Banach space over F under pointwise addition and scalar multiplication and norm defined by

$\| f \| = \sup \; \| [\{f(x, y)\}] \|, \; \| x \| \leq 1, \; \| y \| \leq 1.$

3.0. SEMILINEAR TENSOR PRODUCT :

3. 1. Let X (F_1) and Y (F_2) be linear spaces over arbitrary fields F_1 and F_2 and let Ψ be a bi-semllinear map $\Psi : X (F_1) \times Y (F_2) \to Z (F)$, where Z is a linear space over the field F, F_1 and F_2 being isomorphic to F under isomorphisms ξ_1 and ξ_2 respectively. The couple (Z, Ψ) is called the semilinear tensor product of X and Y if (Z, Ψ) possesses universal factorization property [4] in the sense that for every bi-semilinear map $f : X \times Y \to S (F)$, there exists exists a unique linear map $g : Z \to S$ such that $f = g \circ \Psi$.

3.2. SEMILINEAR TENSOR PRODUCT OF NORMED LINEAR SPACES :

Let X (F_1) and Y (F_2) be normed linear spaces over F_1 and F_2 ; ξ_1, ξ_2 be isomorphisms of F_1 and F_2 to F, which possesses a real valued valuation p. Let X' (F_1) and Y' (F_2) be linear dual spaces of X and Y respectively, i.e. $f : X \to F_1$ and $g : Y \to F_2$ where $f \epsilon X' (F_1)$ and $g \epsilon Y' (F_2)$ and $B L^s (X', Y' ; F)$ denote the space of bounded bi-semilinear maps Ψ : $X' \times Y' \to F$. Let $(X \otimes Y)_F$ denote an element of $BL^s (X', Y' ; F)$ such that

$(X \otimes Y)_F (f, g) = \xi_1 \, f(x) \, \xi_2 \, g(y)$; $f \epsilon X'$, $g \epsilon Y'$.

Then the semilineer tensor product $(X \otimes Y)_F$ is defined to be the linear span of $(x \otimes y)_{\bar{F}}$ in $BL^s (X', Y' ; F)$ for $\tau : X \times Y \to (X \otimes Y)_F$ being a bi-semilinear map. it can be show as below that $(X \otimes Y ; \tau)$ has universal factorization proyerty.

LEMMA 1. Given $u \epsilon (X \otimes Y)_F$, there exist linearly independent sets $\{x_i\}$, $\{y_i\}$ such that

$$u = \sum_{1}^{n} (x_i \otimes y_i)_F$$

PROOF : If possible, let $y_n = \sum_{1}^{n-1} d_i y_i$, $d_i \in F_2$

$$\text{Then } u = \sum_{1}^{n-1} (x_i \otimes y_i)_F + \sum_{1}^{n-1} (x_n \otimes d_i y_i)_F$$

$$= \quad ,, \quad + \sum_{1}^{n-1} \xi_2 (d_i) (x_n \otimes y_i)_F$$

$$= \quad ,, \quad + \sum_{1}^{n-1} \xi_1 (c_i) (x_n \otimes y_i)_F ,$$

$$[\xi_1 (c_i) = \xi_2 (d_i), c_i \in F_1]$$

$$= \sum_{1}^{n-1} (x_i \otimes y_i)_F + \sum_{1}^{n-1} (c_i x_n \otimes y_i)_F$$

$$= \sum_{1}^{n-1} ((x_i + c_i x_n) \otimes y_i)_F$$

which is a contradiction as n is minimal.

Hence y_i and therefore x_i's are linearly independent.

LEMMA : 2. Let $\sum_{1}^{n} (x_i \otimes y_i)_F = 0$, where x_i s are linearly independent. Then

$$y_i = 0, \quad i = 1, 2. \ldots\ldots n.$$

PROOF : We have, for $f \in X' (F_1)$ and $g \in Y' (F_2)$,

$$(\sum_{1}^{n} (x_i \otimes y_i)_F) (f, g) = 0 \in F$$

$$\therefore \sum_{1}^{n} (\xi_1 f (x_i) \, \xi_2 \, g (y_i)) = 0$$

$$\text{i. e. } \xi_1 f (\sum_{1}^{n} (\xi_1^{-1} \{\xi_2 \, g (y_i)\}) x_i) = 0$$

$$\text{i. e. } \sum_{1}^{n} (\xi_1^{-1} \{\xi_2 (g (y_i))\} \; x_i = 0 \in X \text{ [f being arbitrary]}$$

i.e $\xi_1^{-1} \{\xi_2 (g (y_i))\} = 0 \in F_1$ [x_i 's are L. I]

i.e $g(y_i) = 0 \in F_2$, i.e $y_i = 0$ [g being arbitrary]

LEMMA : 3. If $\{x_i\}$; $i=1, 2, \ldots, m$ and $\{y_j\}$; $j=1, 2, \cdots, n$ are linearly independent subsets of X and Y respectively, then $\{x_i \otimes y_j ; i=1, 2, \ldots, m ; j=1, 2, \ldots, n\}$ is a linearly independent subset of $(X \otimes Y)_F$.

Proof is immediate from Lemma 1 and Lemma 2.

Now to prove that $((X \otimes Y)_F ; \tau)$ has Universal factorization property, we consider $\phi : X \times Y \to Z(F)$, any bi-semilinear map and show that there exists a unique linear map $\sigma : ((X \otimes Y)_F) \to Z(F)$ such that $\sigma((x \otimes y)_F) = \phi(x, y)$; $x \in X$, $y \in Y$.

Considering an element of $(X \otimes Y)_F$ as $(\sum_{r=1}^{n} (x_r \otimes y_r))_F$, it is enough to show that $(\sum_{r=1}^{n} (x_r \otimes y_r))_F = 0$ implies $\sum_{r=1}^{n} \phi(x_r, y_r) = 0$, if we claim that σ is defined as $\sigma(\sum_{r=1}^{n} x_r \otimes y_r) = \sum_{r=1}^{n} \phi(x_r, y_r)$. The proof follows directly from Lemma 3.

3.3. Norms on semilinear tensor product spaces :

DEFINITION : 3.3.1 : Let $X(F_1)$ and $Y(F_2)$ be given nermed linear spaces over arbitrary fields F_1 and F_2 which are respectively isomorphic to another field F having a real valued valuation. Then weak norm on $u = \sum_i (x_i \otimes y_i)_F$ is defined by

$$\omega(u) = \sup \| [\{\sum_i \xi_1 f(x_i)\, \xi_2\, g(y_i)\}] \| , \ \|f\| \leqslant 1, \ \|g\| \leqslant 1, \quad f \in X', g \in Y'$$

Obviously $\omega((x \otimes y)_F) = \|x\| \ \|y\|$ [Considering X and Y as linear dual spaces of X′ and Y′ respectively.]

Since $BL^s(X', Y' ; F)$ is a Banach space, therefore $(X \otimes Y)_F$ is closed in $BL^s(X', Y' ; F) \Longleftrightarrow (X \otimes Y)_F$ is complete in $BL^s(X', Y' ; F)$ under ω.

DEFINITION 3.3.2.

The weak semi-linear tensor product of X and Y is defined as the closure of $(X \otimes Y)_F$ in $BL^s(X', Y' ; F)$ [i. e., the completition of $(x \otimes y)_F$ in $BL^s(X', y' ; F)$ and it is denoted by $(X \otimes_\omega Y)_F$.

PROPOSITION 3.1. Let X and Y be two non-empty arbitrary sets and F_1 and F_2 be two arbitrary field of scalars isomorphic to an associative field F (having a real valued valuation) under isomorphisms ξ_1 and ξ_2 respectively. Then there exists a linear isometric isomorphism T′ of $((B(X, F_1) \otimes_\omega B(Y, F_2))_F \to B(X \times Y, F)$ such that

$(T(f \otimes g))' = \xi_1 f(x) \xi_2 g(y)$; $x \in X, y \in Y, f \in B(X, F_1), g \in B(Y, F_2)$ where T is the restriction of T′ to $B(X, F_1) \otimes B(Y, F_2)$.

Proof : We first define a map $\Psi : B(X, F_1) \otimes B(Y, F_2) \rightarrow B(X \times Y; F)$ such that $(\Psi(f, g))(x, y) = \xi_1 f(x) \xi_2 g(y)$; Then Ψ is bi-semilinear as shown below :

$$[\Psi(f+h, g)](x,y) = \xi_1(f+h)(x) \xi_2 g(y)$$
$$= \xi_1[f(x)+h(x)]\xi_2 g(y) \text{ [By definition of } f+g]$$
$$= [\xi_1 f(x) + \xi_1 h(x)] \xi_2 g(y) \text{ [}\xi_1 \text{ being an isomorphism]}$$
$$= \xi_1 f(x) \xi_2 g(y) + \xi_1 h(x) \xi_2 g(y) \text{ [By distributive property of F]}$$
$$= \Psi(f,g)(x, y) + \Psi(h,g)(x,y)$$
$$= [\Psi(f,g) + \Psi(h,g)](x, y) \text{ [By definition of the sum in } B(X \times Y; F)$$

Hence $\Psi[(f+h), g] = \Psi(f, g) + \Psi(h, g)$

Similarly $\Psi(f, g+k) = \Psi(f, g) + \Psi(f, k)$

Also $\Psi(\alpha f, g) = \xi_1(\alpha) \Psi(f, g)$; $\alpha \in F_1$, for

$$[\Psi(\alpha f, g)](x, y) = \xi_1(\alpha f)(x) \xi_2 g(y)$$
$$= \xi_1[\alpha f(x)] \xi_2 g(y) \text{ [By definition of } \alpha f \text{ in } B(X, F_1)]$$
$$= [\xi_1(\alpha) \xi_1(f(x))] \xi_2 g(y) \text{ [}\xi_1 \text{ being an isomorphism]}$$
$$= \xi_1(\alpha) \psi(f, g), \text{ by associative property of F.}$$

Similarly $\psi(f, \beta g) = \xi_2(\beta) \psi(f, g)$; $\beta \in F_2$.

Hence there exists a unique linear mapping $T : (B(X, F_1) \otimes B(Y, F_2))F \rightarrow B(X \times Y; F$; By definition of the tensor product, T is an isomorphism. To show that T is isometric, we consider

$u = \sum_i (f_i \otimes g_i) \in (B(X, F_1) \otimes B(Y, F_2))F$; $f_i \in B(X, F_1)$, $g_i \in B(Y, F_2)$ and prove that $\| T(u) \| = \omega(u)$; infact, denoting norm of the derived field Ω of F by ($\|'$)

$$\| T(u) \| = \| T(\sum_i (f_i \otimes g_i)) \|$$
$$= \sup_{X \times Y} \| [\{T(\sum_i (f_i \otimes g_i))(x, y)\}] \|'$$
$$= \sup_{X \times Y} \| [\{\sum_i \xi_1 f_i(x) \xi_2 g_i(y)\}] \|'$$
$$= \sup_{Y} \| \sum_i \xi_2 g_i(y) f_i \|$$
$$= \sup_{\|\mu\| \leqslant 1} \sup_{Y} \| [\{\sum_i \xi_2 g_i(y) \xi_1 \mu(f_i)\}] \|', \ \mu \in (B(X, F_1))'$$

i. e, $\mu : B(X, F_1) \rightarrow F_1$

$$= \sup_{\|\mu\| \leq 1} \| \sum_i \xi_1 \mu (f_i) g_i \|$$

$$= \sup_{\|\mu\| \leq 1} \sup_{\|\nu\| \leq 1} \| [\{\sum_i \xi_1 \mu (f_i) \xi_2 \nu (g_i)\}] \|', \ \nu \in B(Y, F_2))' \text{ i. e. } \nu : B(Y, F_2) \to F_2$$

$$= \omega (\sum_i f_i \otimes g_i) = \omega (u)$$

Thus T is on isometry of $B(X, F_1) \otimes B(Y, F_2) \to B(X \times Y; F)$; also $B(X \times Y; F)$ is a Banach space, hence there exists an extension $T' : B(X, F_1) \otimes_\omega B(Y, F_2) \to B(X \times Y, F)$.

This proves the proposition.

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