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ON A GENERALIZATION OF HERMITE POLYNOMIALS-I

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Some years ago H. W. Gould and A. T. Hopper [2] introduced a generalization of the usual Hermite polynomials by making the definition

$$(1)\quad H_n^r\ (x,a,p) = (-1)^n\, x^{-a} e^{px^r}\, D^n\left(x^a e^{-px^r}\right),\ D\equiv d/dx,$$

which bears a close relationship with a generalization of the generalized Laguerre polynomials due to the present author [1], viz.

$$(2)\quad H_{kn}^{(\alpha)}\ (x,\ p)=(-1)^n n!\ T_{kn}^{(\alpha-n)}\ (x,\ p),$$

where

$$(3)\quad T_{kn}^{(\alpha)}\ (x,p) = \frac{1}{n\,!}\ x^{-\alpha}\ e^{px^k}\ D^n\ (x^{\alpha+n}\ e^{-px^k})$$

and

$$(4)\quad H_{kn}^{(\alpha)}(x,p)=x^n\ H_n^k\ (x,\ \alpha,\ p),$$

the reason of multiplying H_n^k (x, α, p) by x^n lies in the fact that H_n^k (x, α, p) is not a polynomial, while $H_{kn}^{(\alpha)}$ (x, p) is a polynomial of degre kn, provided k is a natural mumber.

Now from [2, p. 53], we notice that

$$(5)\quad D^k\, H_n^r\ (x,\ a,\ p) = (-1)^k \sum_{j=0}^{k} \binom{k}{j} H_{k-j}^r\ (x,-a,-p)\ H_{n+j}^r\ (x,\ a,\ p).$$

Then operating e^{-tD} on H_n^r (x,a,p), we obtain

$$(6)\quad H_n^r(x-t,a,p)=\sum_{m=0}^{\infty}\frac{t^m}{m!}\sum_{j=0}^{m}\binom{m}{j}H_{m-j}^r(x,-a,-p)\,H_{n+j}^r(x,a,p)$$

$$=\sum_{m=0}^{\infty}\frac{t^m}{m!}H_m^r(x,-a,-p)\sum_{j=0}^{\infty}\frac{t^j}{j!}H_{n+j}^r(x,a,p).$$

The relation (6) can be easily verified by means of the followring two generating relations of Gould—Hopper :

$$(7)\quad \sum_{n=0}^{\infty}\frac{t^n}{n!}H_n^r(x,a,p)=x^{-a}(x-t)^a\,e^{p(x^r-(x-t)^r)}$$

$$(8)\quad \sum_{n=0}^{\infty}\frac{t^n}{n!}H_{n+m}^r(x,a,p)=x^{-a}(x-t)^a\,e^{p(x^r-(x-t)^r)}H_m^r(x-t,a,p)$$

The relation (6), in terms of our polynomials $T_{kn}^{(\alpha)}(x,p)$, reveals that

$$(9)\quad T_{rn}^{(a-n)}(x-t,p)$$

$$=\sum_{m=0}^{\infty}(-xt)^m\,T_{rm}^{(-a-m)}(x,-p)\sum_{j=0}^{\infty}\binom{n+j}{j}(-xt)^j\,T_{r(n+j)}^{(a-n-j)}(x,p)$$

Next from [2, p. 53] we observe that

$$(10)\quad \mathcal{D}^n=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}H_{n-k}^r(x,a,p)\,D^k,$$

where $\mathcal{D}\equiv D-pr\,x^{r-1}+a/x$.

Then operating $e^{-t\mathcal{D}}$ on $H_n^r(x,a,p)$ we obtain

$$(11)\quad x^{-a}(x-t)^a\,e^{p(x^r-(x-t)^r)}H_n^r(x-t,a,p)$$

$$=\sum_{m=0}^{\infty}\frac{t^m}{m!}\sum_{k=0}^{m}(-1)^k\binom{m}{k}H_{m-k}^r(x,a,p)\,D^kH_n^r(x,a,p),$$

by means of the formula of Gould—Hopper :

$$(12)\quad e^{-t\mathcal{D}}f(x)=x^{-a}(x-t)^a\,e^{p(x^r-(x-t)^r)}f(x-t).$$

The relation (11) when combined with (5), yields the relation

$$(13)\quad x^{-a}(x-t)^{a}\, e^{p(x^r-(x-t)^r)}\, H_n^r(x-t, a, p)$$
$$= \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{k=0}^{m} \binom{m}{k} H_{m-k}^r(x,a,p) \sum_{j=0}^{k} \binom{k}{j} H_{k-j}^r(x,-a,-p)\, H_{n+j}^r(x,a,p),$$

which can be easily verified by means of (7) and (8).

Comparing (8) and (13) we get

$$(14)\quad H_{n+m}^r(x,a,p)$$
$$= \sum_{j=0}^{m} \binom{m}{j} H_{n+j}^r(x,a,p) \sum_{k=j}^{m} \binom{m-j}{k-j} H_{m-k}^r(x,a,p)\, H_{k-j}^r(x,-a,-p).$$

It may be of much interest to compare (8) and (11). Indeed, then we have

$$(15)\quad H_{n+m}^r(x,a,p)$$
$$= \sum_{k=0}^{m} (-1)^k \binom{m}{k} H_{m-k}^r(x,a,p)\, D^k H_n^r(x,a,p).$$

Now if we let $r=2$, $a=0$ and $p=1$ in (15) we find the formula for the classical Hermite polynomials

$$(16)\quad H_{n+m}(x) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} H_{m-k}(x)\, D^k H_n(x).$$

Since $D^k H_n(x) = \dfrac{2^k\, n!\, H_{n-k}(x)}{(n-k)!}$, we obtain from (16)

$$(17)\quad H_{n+m}(x) = \sum_{k=0}^{\min(m,n)} (-2)^k \binom{m}{k}\binom{n}{k} k!\, H_{m-k}(x)\, H_{n-k}(x),$$

which is the well-known formula of N. Nielsen.

Next suppose that

$$(18)\quad G(x,t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} H_n^r(x,a,p).$$

Then operating $e^{-t\mathcal{D}}$ on $G(x, tz)$ we obtain by means of (12) and the following formula of Gould-Hopper

$$(19)\quad \mathcal{D}^m H_n^r(x, a, p) = (-1)^m H_{n+m}^r(x, a, p):$$

the bilateral generating relation

$$(20) \quad x^{-a}(x-t)^{a}\, e^{p\,(x^r-(x-t)^r)}\, G\,(x-t,\, t\, z)$$

$$= \sum_{m=0}^{\infty} \frac{(-t)^m}{m!} \sum_{n=0}^{\infty} \frac{a_n\,(tz)^n}{n!}\, \mathcal{D}^m \overset{r}{H}_n\,(x,a,p)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^{m+n}}{m!\,n!}\, a_n\, z^n\, \overset{r}{H}_{n+m}\,(x,a,p)$$

$$= \sum_{m=0}^{\infty} \frac{t^m}{m\,!}\, \overset{r}{H}_m\,(x,a,p) \sum_{n=0}^{m} \binom{m}{n} a_n\, z^n.$$

The above bilateral generating relation can be easily generalized in the form of the follwing theorem :

Theorem. If there exists a generating relation of the form

$$(21) \quad G\,(x,\, t,\, u) = \sum_{n=0}^{\infty} \frac{a_n\, t^n}{n\,!}\, \overset{r}{H}_{n+m}\,(x,a,p)\, q_n\,(u),$$

where $q_n\,(u)$ is any polynomial or function in u, then the following more general generating relation holds

$$(22) \quad x^{-a}(x-t)^{a}\, e^{p\{x^r-(x-t)^r\}}\, G(x-t,\, tz,\, u)$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n\,!}\, \overset{r}{H}_{n+m}\,(x,\, a,\, p) \sum_{k=0}^{n} \binom{n}{k} a_k\, z^k\, q_k\,(u)$$

Proof.

$$\sum_{n=0}^{\infty} \frac{t^n}{n\,!}\, \overset{r}{H}_{n+m}\,(x,a,p) \sum_{k=0}^{n} \binom{n}{k} a_k\, z^k\, q_k\,(u)$$

$$= \sum_{k=0}^{\infty} \frac{(tz)^k}{k\,!}\, a_k\, q_k\,(u) \sum_{n=0}^{\infty} \frac{t^n}{n\,!}\, \overset{r}{H}_{n+m+k}\,(x,\, a,\, p)$$

$$= x^{-a}(x-t)^{a}\, e^{p(x^r-(x-t)^r)} \sum_{k=0}^{\infty} \frac{(tz)^k}{k\,!}\, a_k\, \overset{r}{H}_{m+k}\,(x-t,a,p)\, q_k\,(u) \quad \text{[By (8)]}$$

$$= x^{-a}(x-t)^{a}\, e^{p(x^r-(x-t)^r)}\, G\,(x-t,\, tz,\, u) \quad \text{[By (21)]}.$$

Again returning to the action of the operator $e^{-t\mathcal{D}}$ on G (x, tz) defined by (18) we have by means of (10) and (5)

$$(23) \quad x^{-a}(x-t)^{a}\, e^{p(x^r-(x-t)^r)}\, G\,(x-t,\, tz)$$

$$= \sum_{n=0}^{\infty} \frac{(tz)^n}{n!} a_n \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{k=0}^{m} \binom{m}{k} \overset{r}{H}_{m-k}(x,a,p).$$

$$\cdot \sum_{j=0}^{k} \binom{k}{j} \overset{r}{H}_{k-j}(x,-a,-p) \; \overset{r}{H}_{n+j}(x,a,p),$$

which may be compared with (20) and which can be easily verified by means of (7), (8) and (18).

Lastly considering the action of the operator e^{tD} on $G(x, tz)$ defined by (18), we have by means of (5) and (8)

$$(24) \quad x^a (x+t)^{-a} e^{-p(x^r-(x+t)^r)} G(x+t, tz)$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{p=0}^{n} (-1)^{n-p} \binom{n}{p} a_p z^p \overset{r}{H}_{n-p}(x,-a,-p) \overset{r}{H}_p(x+t, a, p),$$

which can be easily verified by means of (7) and (18). we should like to examine one special case of (24). If we let $a_n=1$ for all n and $z=1$, we obtain from (24) the result

$$(25) \quad \left(\frac{x}{x+t}\right)^{2a} e^{-2p(x^r-(x+t)^r)}$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{p=0}^{n} (-1)^{n-p} \binom{n}{p} \overset{r}{H}_{n-p}(x,-a,-p) \overset{r}{H}_p(x+t, a, p)$$

REFERENCES

[1] Chatterjea, S. K.—On a generalization of Laguerre polynomials, Rend. Sem. Mat, Univ, Padova 34 (1964), 180—190,

[2] Gould, H.W. & Hopper, A.T.—Operational formulas conneeted with two generalizations of Hermite polynomials, Duke Math. J. 29 (1962), 51—64.

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