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ON A GENERALIZATION OF HERMITE POLYNOMIAL-II

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Some years ago H.W. Gould and A. T. Hopper [2] introduced a second generalization of the usual Hermite polynomials by making the definition

(1) $g_n^r(x, h) = e^{hD^r} x^n, \quad D \equiv d/dx,$

a particular case of which was studied by L. R. Bragg [1]

In [1, p. 58] we notice that

(2) $D^j g_n^r(x, h) = j! \binom{n}{j} g_{n-j}^r(x, h).$

Now operating e^{-tD} on $g_n^r(x, h)$ we obtain

$$g_n^r(x-t, h) = \sum_{m=0}^{\infty} \frac{(-t)^m}{m!} D^m g_n^r(x, h)$$

(3) $$= \sum_{m=0}^{n} \binom{n}{m} (-t)^m g_{n-m}^r(x, h),$$

which can be compared with the result (6·19) of Gould-Hopper and which has an interesting special case when $h=-1$ and $r=2$, viz.

(4) $$H_n\left(\frac{x-t}{2}\right) = \sum_{m=0}^{n} \binom{n}{m} (-t)^m H_{n-m}(x/2),$$

which can well be compared with the following result [3, p. 255]

(5) $$H_n(x+y) = \sum_{m=0}^{n} \binom{n}{m} H_m(x) (2y)^{n-m}.$$

Next let

(6) $$G(x, t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} g_n^r(x, h).$$

Then operating e^{-tD} on G (x, t) we get after some calculation

(7) $$G(x-t, t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} \sum_{m=0}^{n} \binom{n}{m} (-t)^m g_{n-m}^r(x, h),$$

which, when compared with (3), inplies that it can be verified easily by means of (3) and (6).

Next we consider the generating series

$$\sum_{m=0}^{\infty} \frac{t^m}{m!} g_{n+m}^r(x, h).$$

Here we have

$$\sum_{m=0}^{\infty} \frac{t^m}{m!} g_{n+m}^r(x, h)$$

$$= \sum_{m=0}^{\infty} \frac{t^m}{m!} e^{hD^r} x^{n+m}$$

$$= e^{hD^r} (x^n e^{tx}).$$

Now from [2, p. 59] we know that

(8) $$e^{D_x^r} (x^n e^{tx}) = D_t^n (e^{ht^r} e^{tx}).$$

Thus we obtain

(9) $$\sum_{m=0}^{\infty} \frac{t^m}{m!} g_{n+m}^r(x, h) = D_t^n (e^{ht^r} e^{tx}).$$

When n=0, we get as special case

(10) $$\sum_{m=0}^{\infty} \frac{t^m}{m!} g_m^r(x, h) = e^{tx+ht^r},$$

which is mentioned in the work of Gould-Hopper. Again when h = –1 and r=2, we obtain from (9) the interesting special case of the usual Hermite polynomials, viz.

(11) $$\sum_{m=0}^{\infty} \frac{t^m}{m!} H_{n+m}(x/2) = D_t^n e^{tx-t^2}).$$

In other words,

$$\sum_{m=0}^{\infty} \frac{t^m}{m!} H_{n+m}(x) = D_t^n \left(e^{2tx-t^2}\right)$$

$$= e^{x^2} D_t^n e^{-(x-t)^2}$$

$$= e^{x^2} . (-1)^n D_\omega^n e^{-\omega^2} \quad [\omega = x-t]$$

$$= e^{x^2} . e^{-\omega^2} H_n(\omega)$$

$$= e^{2xt-t^2} H_n(x-t),$$

which is the well-known form of the generating functon for Hermite polynomials.

Let us now consider the action of e^{tD} on $g_n^r(x, h)$, where $D \equiv x + hr D^{r-1}$.

we have

$$e^{tD} g_n^r(x, h)$$

$$= \sum_{m=0}^{\infty} \frac{t^m}{m!} D^m g_n^r(x, h).$$

Now we know from [2, p. 59]

$$(12) \quad D^m g_n^r(x, h) = g_{n+m}^r(x, h).$$

Thus we obtain

$$(13A) \quad e^{tD} g_n^r(x, h) = \sum_{m=0}^{\infty} \frac{t^m}{m!} g_{n+m}^r(x, h)$$

$$(13B) \quad e^{tD} g_n^r(x, h) = D_t^n \left(e^{tx} e^{ht^r}\right)$$

$$(13C) \quad e^{tD} g_n^r(x, h) = e^{hD_x^r} (x^n e^{tx}).$$

When $h = -1$ and $r=2$, we have the following interesting special cases of the usual Hermite polynomials.

$$(14A) \quad e^{t(2x-D)} H_n(x) = \sum_{m=0}^{\infty} \frac{t^m}{m!} H_{n+m}(x) = e^{2xt-t^2} H_n(x-t)$$

(14B) $e^{t(2x-D)} H_n(x) = D_t^n (2tx - t^2)$

(14C) $e^{t(2x-D)} H_n(x) = e^{-D_x^2/4} (x^n e^{tx})$,

of which (14A) is a special case of the result (5.4) of Gould-Hopper,

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