Jour. Pure Math.
Vol. 2 (1982), pp. 1-4

MY FAVOURITE PROOF OF MEHLER'S FORMULA

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The following formula of Mehler

$$(1)\quad \sum_{n=0}^{\infty} \frac{t^n}{2^n\, n\,!} H_n(x)\, H_n(y) = (1-t^2)^{-1/2} \exp\left[\frac{2xyt-(x^2+y^2)\,t^2}{1-t^2}\right],$$

where $H_n(x)$ is the Hermite polynomial defined by

$$(2)\quad \sum_{n=0}^{\infty} H_n(x)\, t^n/n\,! = \exp\,(2xt-t^2),$$

is well-known. Besides the usual proofs I prefer a proof of Mehler's formula from the linear generating relation (2). My proof is based on some integral formulas, which simultaneously serves as a key to other bilateral or even trilateral (or trilinear) generating relations involving Hermite polynomials.

The said integral formulas are the following

$$(3)\quad H_n(x) \equiv \exp\,(-D^2/4)\,(2x)^n = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2}\,[2(x+iv)]^n\, dv\,;\ D \equiv d/dx$$

$$(4)\quad e^{-x^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2+2ixv}\, dv.$$

Now we have

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} \left(\sum_{k=0}^{\infty} \frac{[2t\,(y+iv)]^k}{k\,!} H_k(x)\right) dv$$

$$= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{t^k}{k\,!} H_k(x) \int_{-\infty}^{\infty} e^{-v^2}\,[2(y+iv)]^k\, dv$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k\,!} H_k(x)\, H_k(y). \qquad \text{(inversion justified)}$$

Thus we can write

$$\sum_{k=0}^{\infty} \frac{t^k}{2^k\, k\,!} H_k(x)\, H_k(y)$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} \left(\sum_{k=0}^{\infty} \frac{[t\,(y+iv)]^k}{k\,!} H_k(x) \right) dv$$

$$(5) \qquad = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\,[-v^2 + 2xt\,(y+iv) - t^2\,(y+iv)^2]\, dv,$$

which is a generating integral equivalent to Mehler's generating series. This method serves a new technique for adjoining a Hermite polynomial to any generating relation involving various special functions of mathematical physics, but it may happen that the generating integral is convergent or divergent. Some divergent generating functions were already seen in the works [3, 4, 5,] of Fred Brafman.

From (5) we obtain finally

$$\sum_{k=0}^{\infty} \frac{t^k}{2^k\, k\,!} H_k(x)\, H_k(y)$$

$$= \frac{\exp\,(2xyt - t^2y^2)}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\,[-v^2\,(1-t^2) + 2it\,(x-yt)\,v]\, dv$$

$$= \frac{\exp\,(2xyt - t^2y^2)}{\sqrt{\pi}\,\sqrt{(1-t^2)}} \int_{-\infty}^{\infty} \exp\left[-w^2 + 2i\,\frac{t\,(x-yt)}{\sqrt{(1-t^2)}}\,w\right] dw$$

$$= \frac{\exp\,(2xyt - t^2y^2)}{\sqrt{(1-t^2)}} .\ \exp\left[-\frac{t^2(x-yt)^2}{1-t^2}\right]$$

$$= (1-t^2)^{-1/2} \exp\left[\frac{2xyt - (x^2+y^2)t^2}{1-t^2}\right],$$

which is (1).

Similarly starting from the generating relation

$$(6) \qquad \sum_{n=0}^{\infty} \frac{H_{n+k}(x)\, t^n}{n\,!} = \exp\,(2xt - t^2)\, H_k\,(x-t),$$

we derive

$$\sum_{n=0}^{\infty} \frac{t^n}{2^n\, n!} H_{n+k}(x)\, H_n(y)$$

$$(7) \quad = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left[-v^2 + 2xt\,(y+iv) - t^2\,(y+iv)^2\right] H_k\,(x - t\,(y+iv))\, dv,$$

which is an equivalent generating integral for the generating series. We know that the generating function for the generating series is

$$(1-t^2)^{-\frac{1}{2}(k+1)} \exp\left[\frac{2xyt-(x^2+y^2)\,t^2}{1-t^2}\right] H_k\left(\frac{x-yt}{(1-t^2)^{1/2}}\right).$$

Thus we obtain the following integral formula

$$(8) \quad \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left[-v^2\,(1-t^2) + 2it\,(x-yt)\,v\right] H_k\,(x - t\,(y+iv))\, dv$$

$$= (1-t^2)^{-(k+1)/2} \exp\left[-\frac{t^2\,(x-yt)^2}{1-t^2}\right] H_k\left(\frac{x-yt}{(1-t^2)^{1/2}}\right),$$

which does not seem to appear before.

Now it may be of interest to point out that starting from the generating relation of L. Carlitz [6]

$$(9) \quad \sum_{k=0}^{\infty} \frac{t^k}{2^k\, k!} H_{k+m}(x)\, H_{k+n}(y)$$

$$= (1-t^2)^{-(m+n+1)/2} \exp\left[\frac{2xyt-(x^2+y^2)\,t^2}{1-t^2}\right].$$

$$\sum_{r=0}^{\min(m,n)} 2^r\, r!\, \binom{m}{r} \binom{n}{r}\, t^r\, H_{m-r}\left(\frac{x-ty}{(1-t^2)^{1/2}}\right) H_{n-r}\left(\frac{y-xt}{(1-t^2)^{1/2}}\right),$$

we derive in the same manner

$$(10) \quad \sum_{k=0}^{\infty} \frac{(t/4)^k}{k!} H_k(x)\, H_{k+m}(y)\, H_{k+n}(z)$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2}\,(1-t^2\,(x+iv)^2)^{-(m+n+1)/2}$$

$$\cdot \exp\left[\frac{2yzt(x+iv)-(y^2+z^2)t^2\,(x+iv)^2}{1-t^2\,(x+iv)^2}\right].$$

$$\cdot \sum_{r=0}^{\min(m,n)} 2^r\, r!\, \binom{m}{r} \binom{n}{r} (t(x+iv))^r\, H_{m-r}\left(\frac{y-t(x+iv)z}{(1-t^2(x+iv)^2)^{1/2}}\right)$$

$$. H_{n-r}\left(\frac{z-t(x+iv)y}{(1-t^2(x+iv)^2)^{1/2}}\right) dv.$$

A particular case of (10) is worthy of much notice. Indeed, using $m=n=o$ we obtain the following trilinear generating integral for the Hermite polynomials

$$(11) \qquad \sum_{k=0}^{\infty} \frac{(t/4)^k}{k!} H_k(x)\, H_k(y)\, H_k(z)$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} (1-t^2(x+iv)^2)^{-1/2} \exp\left[\frac{2yzt(x+iv)-t^2(y^2+z^2)(x+iv)^2}{1-t^2(x+iv)^2}\right] dv,$$

which may be compared with the remark made by R. Askey [2] and W. A. Al-Salam and L. Carlitz [1] in connection with the trilinear generating function for the Hermite polynomials.

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Received
2. 1. 1982

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