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ON A GENERATING FUNCTION OF FENG

S. K. CHATTERJEA

In a recent paper [2], C. C. Feng has derived the following main generating relation involving modified Laguerre polynomials

$$(1)\quad exp(-a_{23}xz)exp\left(-\frac{a_{12}}{y}(1+a_{13}y+a_{23}z)(1+a_{13}y+a_{23}z)^{-n-\beta}\right.$$

$$\cdot f_n^{(\beta)}\left[\left(\frac{a_{12}}{y}+\frac{a_{22}}{z}+x\right)\left(1+a_{13}y+a_{23}z\right)\right]$$

$$=\sum_{p=0}^{\infty}\frac{(a_{23})^p}{p!}\sum_{m=0}^{\infty}\frac{(a_{13})^m}{m!}\sum_{l=0}^{\infty}\frac{(a_{22})^l}{l!}\sum_{k=0}^{\infty}\frac{(a_{12})^k}{k!}(-1)^{k+m+p}(\beta-k)_m(n-l+1)_p.$$

$$\cdot f_{n-l+p}^{(\beta-k+m)}(x)y^{m-k}z^{p-l},$$

by replacing β by $y\frac{\partial}{\partial y}$, n by $z\frac{\partial}{\partial z}$ and $f_n^{(\beta)}(x)$ by $u(x, y, z)$ in the linear differential relation

$$(2)\quad xD^2 f_n^{(\beta)}(x)+(1-x-n-\beta)\, Df_n^{(\beta)}(x)+nf_n^{(\beta)}(x)=0,$$

and by following the method of L. Weisner [3].

It may be of interest to remark that the result (1) of Feng follows easily with the help of the following unilateral generating relations :

$$(3)\quad e^{xt}(1-t)^{-\beta-v} f_v^{(\beta)}(x(1-t)) = \sum_{n=0}^{\infty}\frac{(v+1)_n}{n!} f_{v+n}^{(\beta)}(x)t^n$$

$$(4)\quad f_n^{(\beta)}(x+t) = \sum_{n=0}^{v}\frac{1}{n!} f_{v-n}^{(\beta)}(x)t^n$$

$$(5)\quad (1-t)^{-\beta-n} f_n^{(\beta)}((1-t)x) = \sum_{m=0}^{\infty}\frac{t^m}{m!}(\beta)_m f_n^{(\beta+m)}(x)$$

$$(6)\quad e^t f_n^{(\beta)}(x-t) = \sum_{k=0}^{\infty}\frac{t^k}{k!} f_n^{(\beta-k)}(x).$$

Furthermore using the following result of our work [1]

$$(7)\quad e^{y}(wy-1)^{\alpha} L_n^{(\alpha)}\left(\frac{(x-y)(wy-1)}{wy}\right)=\sum_{m=0}^{\infty}\frac{y^m}{m!}\sum_{p=0}^{\infty}\frac{(-n-\alpha)_p (wy)^{\alpha-p}}{p!} L_n^{(\alpha+m-p)}(x)$$

we can deduce a relation analogous to (1) by means of (3) and (4), viz.

$$(8)\quad \sum_{p=0}^{s}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\beta)_p (n-k+1)_l y^{m-p} t_1^l t_2^k}{m!\,p!\,l!\,k!} f_{n-k+l}^{(\beta-m+p)}(x)$$

$$= exp(y+t_1(x-y))(1-t_1-y^{-1})^{-\beta-n} f_n^{(\beta)}[(1-t_1-y^{-1})(x-y+t_2)].$$

The very nature of $(\beta)_p$ in the left member of (8) implies that another result analogous to (1) can be put in the form

$$(9)\quad \sum_{p=0}^{\infty}\frac{(a_{23})^p}{p!}\sum_{m=0}^{\infty}\frac{(a_{13})^m}{m!}\sum_{l=0}^{\infty}\frac{(a_{22})^l}{l!}\sum_{k=0}^{\infty}\frac{(a_{12})^k}{k!}(-1)^{k+m+p}(\beta)_m(n-l+1)_p.$$

$$\cdot f_{n-l+p}^{(\beta-k+m)}(x)\, y^{m-k} z^{p-l}$$

$$= exp\left[-a_{23}xz-\frac{a_{12}}{y}(1-a_{23}z)\right](1-a_{13}y-a_{23}z)^{-n-\beta}.$$

$$\cdot f_n^{(\beta)}\left[(1-a_{13}y-a_{23}z)\left(x+\frac{a_{12}}{y}+\frac{a_{22}}{z}\right)\right].$$

Now the generating relations (3) and (4) are mentioned in [3, p. 45].

Also the generating relations (5) and (6) can be easily deduced from the results of the present author [1]. In fact, in [1, p. 369] we notice that

$$(10)\quad (1-t)^{\alpha} L_n^{(\alpha)}(x(1-t)) = \sum_{m=0}^{\infty}\frac{(-\alpha-n)_m}{m!} L_n^{(\alpha-m)}(x)\, t^m.$$

To prove (5) we make use of the relation

$$(11)\quad f_n^{(\beta)}(x) = (-1)^n L_n^{(-\beta-n)}(x),$$

so that we have

$$\sum_{m=0}^{\infty}\frac{t^m}{m!}(\beta)_m f_n^{(\beta+m)}(x)$$

$$= (-1)^n \sum_{m=0}^{\infty}\frac{t^m}{m!}(\beta)_m L_n^{(-\beta-n)-m}(x)$$

$$= (-)^n (1-t)^{-\beta-n} L_n^{-\beta-n}(x(1-t))$$

$$= (1-t)^{-\beta-n} f_n^{(\beta)}(x(1-t)).$$

Again we notice that [1, p. 370]

$$(12)\quad e^{y}\, L_n^{(\alpha)}(x-y) = \sum_{m=0}^{\infty} \frac{y^m}{m!}\, L_n^{(\alpha+m)}(x).$$

Thus we can prove (6) as follows

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} f_n^{(\beta-k)}(x) = (-1)^n \sum_{k=0}^{\infty} \frac{t^k}{k!} L_n^{(-\beta-n)+k}(x)$$

$$=(-1)^n e^t\, L_n^{(-\beta-n)}(x-t) = e^t\, f_n^{(\beta)}(x-t).$$

We are now in a position to prove the result (1) of Feng in a quite easy manner. Indeed, we have by virtue of (3)

$$\sum_{p=0}^{\infty} \frac{(a_{23})^p}{p!} \sum_{m=0}^{\infty} \frac{(a_{13})^m}{m!} \sum_{l=0}^{\infty} \frac{(a_{22})^l}{l!} \sum_{k=0}^{\infty} \frac{(a_{12})^k}{k!} (-1)^{k+m+p} (\beta-k)_m (n-l+1)_p .$$

$$\cdot f_{n-l+p}^{(\beta-k+m)}(x)\, y^{m-k} z^{p-l}$$

$$= e^{-a_{23}xz} (1+a_{23}z)^{-\beta-n} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-a_{12}/y)^k}{k!} \frac{(-a_{13}y)^m}{m!} \frac{(a_{22}/z)^l}{l!}.$$

$$\cdot (\beta-k)_m (1+a_{23}z)^{k-m+l} f_{n-l}^{(\beta-k+m)}(x(1+a_{23}z))$$

Next using (4) we obtain

Right member of (1)

$$= e^{-a_{23}xz} (1+a_{23}z)^{-\beta-n} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{[-a_{12}(1+a_{23}z)]^k}{k!} \frac{\left(\frac{-a_{13}y}{1+a_{23}z}\right)^m}{m!}.$$

$$\cdot (\beta-k)_m\, f_n^{(\beta-k+m)}\left((1+a_{23}z)\left(x+\frac{a_{22}}{z}\right)\right)$$

Again using (5) we derive

Right member of (1)

$$= e^{-a_{23}xz} (1+a_{13}y+a_{23}z)^{-\beta-n} \sum_{k=0}^{\infty} \frac{(-a_{12}(1+a_{13}y+a_{23}z)/y)^k}{k!}.$$

$$\cdot f_n^{(\beta-k)}\left(\left(x+\frac{a_{22}}{z}\right)(1+a_{13}y+a_{23}z)\right)$$

Lastly using (6) we obtain

Right member of (1)

$$= exp\ (-a_{23}\ xz)\ exp\ \left(-\frac{a_{12}}{y}\ (1+a_{13}\ y+a_{23}\ z)\right)\ (1+a_{13}\ y+a_{23}\ z)^{-\beta-n}.$$

$$.f_n^{(\beta)}\left[\left(1+a_{13}\,y+a_{23}\,z\right)\left(x+\frac{a_{12}}{y}+\frac{a_{23}}{z}\right)\right],$$

which is the left member of (1).

Next we consider the result (7). It follows from (7) by virtue of (11)

$$\sum_{m=0}^{\infty} \frac{y^m}{m\,!} \sum_{p=0}^{\infty} \frac{(\beta)_p\ (y)^{-p}}{p\,!}\ f_{n-k+l}^{(\beta-m+k)}\ (x)$$

(13)

$$= e^y \left(\frac{y-1}{y}\right)^{-\beta-n+k-l} f_{n-k+l}^{(\beta)} \left(\frac{(x-y)\ (y-1)}{y}\right).$$

Thus

$$\sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\beta)_p\ (n-k+1)_l\ y^{m-p}\ t_1^l\ t_2^k}{m\,!\ p\,!\ l\,!\ k\,!}\ f_{n-k+l}^{(\beta-m+p)}\ (x)$$

$$= e^y \left(\frac{y-1}{y}\right)^{-\beta-n} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(n-k+1)_l}{l\,!\ k\,!} \left(\frac{t_1\,y}{y-1}\right)^l \left(\frac{t_2\,(y-1)}{y}\right)^k.$$

$$.f_{n-k+l}^{(\beta)}\left(\frac{(x-y)(y-1)}{y}\right)$$

Using the relations (3) and (4) successively we obtain (8) which is analogous to (1).

Finally if we consider the series

$$\sum_{p=0}^{\infty} \frac{(a_{23})^p}{p\,!} \sum_{m=0}^{\infty} \frac{(a_{13})^m}{m\,!} \sum_{l=0}^{\infty} \frac{(a_{23})^l}{l\,!} \sum_{k=0}^{\infty} \frac{(a_{12})^k}{k\,!}\ (-1)^{k+m+p}\ (\beta)_m\ (n-l+1)_p.$$

$$.\ f_{n-l+p}^{(\beta-k+m)}\ (x)\ y^{m-k}\ z^{p-l}$$

and first sum the series over k, then we obtain

$$e^{-a_{12}/y}\sum_{p=0}^{\infty}\frac{(a_{23})^p}{p\,!}\sum_{m=0}^{\infty}\frac{(a_{13})^m}{m\,!}\sum_{l=0}^{\infty}\frac{(a_{22})^l}{l\,!}(-1)^{m+p}(\beta)_m(n-l+1)_p\cdot$$

$$\cdot f_{n-l+p}^{(\beta+m)}\left(x+\frac{a_{12}}{y}\right)y^m z^{p-l}$$

Then we sum the above triple series by means of (3), (4) and (5) and obtain (9).

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Dept. of Pure Math.
Calcutta University