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ON ORDER AND TYPE OF AN ENTIRE FUNCTION REPRESENTED BY DOUBLE DIRICHLET SERIES

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1. Introduction : The growth properties of an entire function represented by Dirichlet series in one complex variable have been studied by a number of mathematicians. But the study of the growth properties of an entire function in several variables represented by multiple Dirichlet series has not yet been done adequately. The main purpose of this paper is to extend the concepts of order and type of an entire function represented by Dirichlet series in one complex variable to an entire function in two variables represented by double Dirichlet series and to express them in terms of co-efficients and exponents. We also construct some entire double Dirichlet series with given pair of positive integers as an order point and with given type and discuss the convexity of a few sets involving the order of entire double Dirichlet series.

Consider the double Dirichlet series

$$(1.1)\quad f(s_1, s_2)=\sum_{m,n=1}^{\infty} a_{mn} \exp (s_1\lambda_m+s_2\mu_n)(s_j=\sigma_j+i\tau_j,\ j=1, 2)$$

where $a_{mn} \in c$, the field of complex numbers, λ_m's, μ_n's are

real, $0 \leqslant \lambda_1 < \lambda_2 < \ldots\ldots < \lambda_m \to \infty$, $0 \leqslant \mu_1 < \mu_2 < \ldots\ldots < \mu_n \to \infty$.

A. I. Janusanskas in his paper [4] had shown that if

$$(1.2)\quad \lim_{m\to\infty} \frac{\log m}{\lambda_m}=0,\ \lim_{n\to\infty} \frac{\log n}{\mu_n}=0$$

then the domain of convergence of the series (1.1) coincides with its domain of absolute convergence. P. K. Sarkar in his paper [5] had shown that the necessary and sufficient condition that the series (1·1) satisfying (1·) to be entire is that

$$(1.3)\quad \lim_{(m,n)\to\infty} \frac{\log | a_{mn} |}{\lambda_m+\mu_n}=-\infty .$$

2. Definitions and notations : We indicate the elements (s_1, s_2) (Res_1, Res_2) etc. of c^2 by their corresponding unsuffixed symbols s, Res etc.

For $(p, r) \in R^2$ (2 dimensional Euclidean space) we say that,

(I) $p \leq r \Leftrightarrow p_j \leqslant r_j, j = 1, 2$

(II) $p < r \Leftrightarrow p \leqslant r$ but $p \neq r$

(III) $p \ll r \Leftrightarrow p_j < r_j, j = 1, 2.$

Let F stand for the family of all double Dirichlet series of the form (1.1) satisfying (1.2) and (1.3). Then $f \in F$ denotes an entire function over c^2.

Corresponding to a $f \in F$ we define the functions : the maximum modulus $M = M_f$ and the maximum term $\mu = \mu_f$ on R^2 by

$$M(\sigma) = M_f(\sigma) = \max \{ | f(s) | : s \in c^2, \operatorname{Re} s = \sigma \}$$

$$\mu(\sigma) = \mu_f(\sigma) = \max_{(m,n) \in N^2} \{ | a_{mn} | \exp (\sigma_1 \lambda_m + \sigma_2 \mu_n) \}$$

where N is the set of natural numbers.

We define the product order and type of an entire Dirichlet series over c^2 in the following way.

Let $f \in F$ and $P_f \subset R^2$ be the set of points $\alpha \in R^2$ such that for every $\alpha \in P_f$ there exists a $\sigma^0 = (\sigma_1^0, \sigma_2^0)$ such that

$$\log M(\sigma) \leqslant \exp (\sigma_1 \alpha_1 + \sigma_2 \alpha_2) \text{ for } \sigma \geqslant \sigma^0, \sigma \in R^2$$

The closure $\bar{P}_f$ of the set P_f is called the product order set of f. The boundary ∂P_f of the set P_f is called the product order of f. A point $\rho \in \partial P_f$ is called a product order point of f. We say that f is of infinite or finite product order according as P_f is empty or non-empty. Evidently for any

$$\rho \in \partial P_f, \rho \geqslant \bar{o} = (o, o).$$

For brevity hence forth we shall call product order simply as order throughout this paper.

It follows from the definition that the set P_f satisfies the following condition :

If $\alpha \in \partial P_f$ then $\{\alpha' : \alpha' \in R^2, \alpha' \gg \alpha\} \subset P_f$ and

if $\alpha \in \partial P_f$ then $\{\alpha' : \alpha' \in R^2, \alpha' \ll \alpha\} \cap P_f = \phi$

Next let us take an order point $\rho\ (\gg \bar{o}) \in \partial P_f$ and denote by $T_f\ (\rho)$ the set of all $T \in R$ such that

$$\log M(\sigma) \leqslant T \exp (\sigma_1 \rho_1 + \sigma_2 \rho_2) \quad \text{for} \quad \sigma \geqslant \sigma^{(1)}, \sigma \in R^2$$

Then the infimum of the values of T for which the above relation is satisfied is called the type of f w. r. t the order point ρ. f is said to be of normal type if type of f is finite and $> o$ and of minimum type if type of $f = 0$. Also, f is said to be of infinite or (max) type if $T_f(\rho)$ is empty.

Theorem 2.1: Let $f \in F$. Then $\rho = (\rho_1, \rho_2) \in R^2_+$ is a product order point of f if

$$\limsup_{(m,n)\to\infty} \frac{\frac{\lambda_m}{\rho_1} \log \lambda_m + \frac{\mu_n}{\rho_2} \log \mu_n}{-\log |a_{mn}|} = 1.$$

Proof: Let us suppose that $0 < \epsilon < 1$. Then we have two sequences $\{\lambda_{m_p}\}$ and $\{\mu_{n_q}\}$ with $m_p \to \infty$ as $p \to \infty$ and $n_q \to \infty$ as $q \to \infty$ such that

$$\log |a_{mn}| > -(1-\epsilon)^{-1} \left\{\frac{\lambda_m}{\rho_1} \log \lambda_m + \frac{\mu_n}{\rho_2} \log \mu_n\right\} \text{for } m = m_p \text{ and } n = n_q$$

Since the inequality $M(\sigma_1, \sigma_2) \geqslant |a_{mn}| \exp(\sigma_1\lambda_m + \sigma_2\mu_n)$ holds for all σ_1, σ_2 and m, n it follows that for all σ_1 and σ_2 and $m = m_p$ and $n = n_q$ that

$$\log M(\sigma_1, \sigma_2) > \log |a_{mn}| + \sigma_1\lambda_m + \sigma_2\mu_n$$

$$> -(1-\epsilon)^{-1}\left[\frac{\lambda_m}{\rho_1} \log \lambda_m + \frac{\mu_n}{\rho_2} \log \mu_n\right] + \sigma_1\lambda_m + \sigma_2\mu_n$$

$$= \frac{\lambda_m}{\rho_1}\left[\sigma_1\rho_1 - (1-\epsilon)^{-1} \log \lambda_m\right] + \frac{\mu_n}{\rho_2}\left[\sigma_2\rho_2 - (1-\epsilon)^{-1} \log \mu_n\right]$$

Taking $\sigma_1\rho_1 = (1-\epsilon)^{-1} \log (e\,\lambda_m)$

$\sigma_2\rho_2 = (1-\epsilon)^{-1} \log (e\,\mu_n)$

we have $\sigma_1\rho_1 - (1-\epsilon)^{-1} \log \lambda_m = (1-\epsilon)^{-1}$

$\sigma_2\rho_2 - (1-\epsilon)^{-1} \log \mu_n = (1-\epsilon)^{-1}$

and $e^{\sigma_1\rho_1(1-\epsilon)} = e\,\lambda_m$

$e^{\sigma_2\rho_2(1-\epsilon)} = e\,\mu_n$

$$\therefore \frac{e^{\sigma_1\rho_1(1-\epsilon)}}{e\,\rho_1} = \frac{\lambda_m}{\rho_1} \text{ and } \frac{e^{\sigma_2\rho_2(1-\epsilon)}}{e\,\rho_2} = \frac{\mu_n}{\rho_2}$$

$$\therefore \log M(\sigma_1, \sigma_2) > \frac{e^{\sigma_1\rho_1(1-\epsilon)}}{e\,\rho_1(1-\epsilon)} + \frac{e^{\sigma_2\rho_2(1-\epsilon)}}{e\,\rho_2(1-\epsilon)} > \frac{e^{\sigma_1\rho_1(1-\epsilon)}}{e\,\rho_1(1-\epsilon)}$$

$$\therefore \quad \log\log M(\sigma) > \sigma_1\rho_1(1-\epsilon) - \log[e\,\rho_1(1-\epsilon)]$$

$$\therefore \quad \frac{\log\log M(\sigma)}{\sigma_1\rho_1+\sigma_2\rho_2} > \frac{\sigma_1\rho_1(1-\epsilon)}{\sigma_1\rho_1+\sigma_2\rho_2} - \frac{\log[e\,\rho_1(1-\epsilon)]}{\sigma_1\rho_1+\sigma_2\rho_2}$$

$$\therefore \quad \limsup_{(\sigma_1,\sigma_2)\to\infty} \frac{\log\log M(\sigma)}{\sigma_1\rho_1+\sigma_2\rho_2} \geqslant \limsup_{(\sigma_1,\sigma_2)\to\infty} \frac{\sigma_1\rho_1(1-\epsilon)}{\sigma_1\rho_1+\sigma_2\rho_2} = 1 \text{ (See [1])} \qquad \ldots \text{(A)}$$

Again, we see that

$$\frac{\frac{\lambda_m}{\rho_1}\log\lambda_m + \frac{\mu_n}{\rho_2}\log\mu_n}{-\log|a_{mn}|} < 1+\epsilon \qquad \text{for } m > m_0,\ n > n_0$$

$$\therefore \quad \lambda_m^{-\frac{\lambda_m}{\rho_1}} \cdot \mu_n^{-\frac{\mu_n}{\rho_2}} > |a_{mn}|^{1+\epsilon} \qquad \text{for } m > m_0,\ n > n_0$$

$$\therefore \quad |a_{mn}| < \lambda_m^{-\frac{\lambda_m}{\rho_1(1+\epsilon)}} \cdot \mu_n^{-\frac{\mu_n}{\rho_2(1+\epsilon)}} \qquad \text{for } m > m_0,\ n > n_0$$

$$\text{Now } M(\sigma_1,\sigma_2) \leqslant \left[\sum_{m=1}^{m_0}\sum_{n=1}^{n_0} + \sum_{m=m_0+1}^{\infty}\sum_{n=1}^{n_0} + \sum_{m=1}^{m_0}\sum_{n=n_0+1}^{\infty} + \sum_{m=m_0+1}^{\infty}\sum_{n=n_0+1}^{\infty}\right] |a_{mn}| \exp(\sigma_1\lambda_m+\sigma_2\mu_n)$$

$$= \sum_1 + \sum_2 + \sum_3 + \sum_4 \text{ (say)}$$

clearly

$$\sum_1 = O[\exp(\sigma_1\lambda_{m_0}+\sigma_2\mu_{n_0})]$$

$$\sum_4 < \sum_{m=m_0+1}^{\infty}\sum_{n=n_0+1}^{\infty} \lambda_m^{-\frac{\lambda_m}{\rho_1(1+\epsilon)}},\ \mu_n^{-\frac{\mu_n}{\rho_2(1+\epsilon)}} \exp(\sigma_1\lambda_m+\sigma_2\mu_n)$$

$$= \sum_{m>m_0}\sum_{n>n_0} \exp\left[\sigma_1\lambda_m+\sigma_2\mu_n-(1+\epsilon)^{-1}\left(\frac{\lambda_m}{\rho_1}\log\lambda_m+\frac{\mu_n}{\rho_2}\log\mu_n\right)\right]$$

$$= \sum_{m>m_0}\sum_{n>n_0} \exp\left[\sigma_1\lambda_m+\sigma_2\mu_n-\frac{\frac{\lambda_m}{\rho_1}\log\lambda_m+\frac{\mu_n}{\rho_2}\log\mu_n}{1+2\epsilon}\left(1+\frac{\epsilon}{1+\epsilon}\right)\right]$$

$$= \sum_{m>m_0}\sum_{n>n_0} \exp\left[\sigma_1\lambda_m+\sigma_2\mu_n-\frac{\frac{\lambda_m}{\rho_1}\log\lambda_m+\frac{\mu_n}{\rho_2}\log\mu_n}{1+2\epsilon} - \frac{\frac{\lambda_m}{\rho_1}\log\lambda_m+\frac{\mu_n}{\rho_2}\log\mu_n}{\epsilon^{-1}(1+\epsilon)(1+2\epsilon)}\right]$$

$$(2.1)\quad \leqslant \max_{(\lambda_m,\, \mu_n)} \exp\left[\sigma_1\lambda_m+\sigma_2\mu_n-(1+2\epsilon)^{-1}\left(\frac{\lambda_m}{\rho_1}\log\lambda_m+\frac{\mu_n}{\rho_2}\log\mu_n\right)\right]$$

$$\times \sum_{m>m_0}\sum_{n>n_0}\exp\left\{\frac{-\frac{\lambda_m}{\rho_1}\log\lambda_m-\frac{\mu_n}{\rho_2}\log\mu_n}{\epsilon^{-1}(1+\epsilon)(1+2\epsilon)}\right\}$$

Since the maximum of the expression

$\exp\left[\sigma_1\lambda_m+\sigma_2\mu_n-(1+2\epsilon)^{-1}\left(\frac{\lambda_m}{\rho_1}\log\lambda_m+\frac{\mu_n}{\rho_2}\log\mu_n\right)\right]$ is attained at

$\lambda_m=e^{-1}\exp\{\sigma_1\rho_1(1+2\epsilon)\}$ and $\mu_n=e^{-1}\exp\{\sigma_2\rho_2(1+2\epsilon)\}$ and the maximum value of the expression is

$$\exp\left[\frac{1}{e\rho_1(1+2\epsilon)}\exp\left\{\sigma_1\rho_1(1+2\epsilon)\right\}+\frac{1}{e\rho_2(1+2\epsilon)}\exp\left\{\sigma_2\rho_2(1+2\epsilon)\right\}\right]$$

$$\leqslant \exp\left[\frac{1}{e\rho(1+2\epsilon)}\left\{\exp\sigma_1\rho_1(1+2\epsilon)+\exp\sigma_2\rho_2(1+2\epsilon)\right\}\right]$$

where $\rho=\min(\rho_1,\rho_2)$

Since the series on the right of (2.1) is convergent,

$$\therefore\ \sum_4 < A\exp\left[\frac{1}{e\rho(1+2\epsilon)}\left\{\exp\left(\sigma_1\rho_1(1+2\epsilon)\right)+\exp\sigma_2\rho_2(1+2\epsilon)\right\}\right]$$

where A is an absolute constant.

Further to estimate Σ_2 it is noted that for all values of m and n, $\exists$ some tive const. k s. t

$$\frac{\log \lambda_m^{\frac{\lambda_m}{\rho_1}}\cdot\mu_n^{\frac{\mu_n}{\rho_2}}}{-\log|a_{mn}|}\leqslant k.$$

Therefore,

$$\Sigma_2\leqslant\sum_{m>m_0}\sum_{n=1}^{n_0}\exp\left[(\sigma_1\lambda_m+\sigma_2\mu_n)-\frac{k^{-1}\lambda_m\log\lambda_m}{\rho_1}-\frac{k^{-1}\mu_n\log\mu_n}{\rho_2}\right]$$

$$=O\left[\exp(\sigma_2\mu_{n_0})\right]\sum_{m>m_0}\left[\exp\left(\sigma_1\lambda_m-\frac{k^{-1}\lambda_m\log\lambda_m}{\rho_1}\right]\right.$$

$$\leqslant O\left[\exp(\sigma_2\mu_{no})\right]. \max_{\lambda_m}\left[\exp\left(\sigma_1\lambda_m - \frac{(k+\epsilon)^{-1}.\lambda_m\log\lambda_m}{\rho_1}\right)\right]$$

$$\times \sum_{m>m_0}\exp\left[-\epsilon k^{-1}(k+\epsilon)^{-1}.\frac{\lambda_m\log\lambda_m}{\rho_1}\right]$$

But the maximum of the expression $\exp\left[\sigma_1\lambda_m - \frac{(k+\epsilon)^{-1}\lambda_m\log\lambda_m}{\rho_1}\right]$

is attained at $\lambda_m = e^{-1}\exp[\sigma_1\rho_1(k+\epsilon)]$

Hence $\sum_2 < O\left[\exp(\sigma_2\mu_{n_0})\right]\exp\left[\frac{e^{-1}(k+\epsilon)^{-1}}{\rho_1}\exp\left\{\sigma_1\rho_1(k+\epsilon)\right\}\right]$

since $\sum_{m>m_0}\exp\left[-\epsilon k^{-1}(k+\epsilon)^{-1}\frac{\lambda_m\log\lambda_m}{\rho_1}\right]$ is convergent.

Similarly $\sum_3 \leqslant O\left[\exp(\sigma_1\lambda_{mo})\right]\exp\left[\frac{e^{-1}(k+\epsilon)^{-1}}{\rho_2}\exp\left\{\sigma_2\rho_2(k+\epsilon)\right\}\right]$

Substituting these values of $\sum_i (1 \leqslant i \leqslant 4)$ in M (σ_1, σ_2)

$$M(\sigma_1,\sigma_2) < \sum_1 + \sum_2 + \sum_3 + \sum_4$$

$$< O\left[\exp(\sigma_1\lambda_{mo}+\sigma_2\mu_{no})\right] + O\left[\exp(\sigma_2\mu_{no})\right]\exp\left[\frac{e^{-1}(k+\epsilon)^{-1}}{\rho_1}\exp\sigma_1\rho_1(k+\epsilon)\right]$$

$$+O\left[\exp(\sigma_1\lambda_{mo})\right]\exp\left[\frac{e^{-1}(k+\epsilon)^{-1}}{\rho_2}\exp\left\{\sigma_2\rho_2(k+\epsilon)\right\}\right]$$

$$+A\exp\left[\frac{1}{e\rho'(1+2\epsilon)}\left\{\exp\left(\sigma_1\rho_1(1+2\epsilon)\right)+\exp\left(\sigma_2\rho_2(1+2\epsilon)\right)\right\}\right]$$

$$= A\exp\left\{\frac{1}{e\rho'(1+2\epsilon)}\left[\exp\sigma_1\rho_1(1+2\epsilon)+\exp\sigma_2\rho_2(1+2\epsilon)\right]\right\}[1+O(1)]$$

$\therefore$ $\log M(\sigma_1,\sigma_2) \leqslant \log A$

$$+\frac{1}{e\rho'(1+2\epsilon)}\left[\exp\sigma_1\rho_1(1+2\epsilon)+\exp\sigma_2\rho_2(1+2\epsilon)\right] \quad \text{for } \sigma > \sigma^0$$

$$= \frac{1}{e\rho'(1+2\epsilon)}\left[\exp\sigma_1\rho_1(1+2\epsilon)+\exp\sigma_2\rho_2(1+2\epsilon)\right] \quad \text{for } \sigma > \sigma^0.$$

$$\therefore \quad \log\log M(\sigma_1, \sigma_2) \leqslant \log \frac{1}{e\rho'(1+2\epsilon)} + \log\left[e^{\sigma_1\rho_1(1+2\epsilon)} + e^{\sigma_2\rho_2(1+2\epsilon)}\right]$$

for $\sigma > \sigma^0$.

$$\leqslant \log \frac{1}{e\rho'(1+2\epsilon)} + \log 2 \,.\, e^{\sigma_1\rho_1(1+2\epsilon)} \text{ if } \rho_1 > \rho_2.$$

$$\therefore \quad \limsup_{(\sigma_1, \sigma_2)\to\infty} \frac{\log\log M(\sigma_1, \sigma_2)}{\sigma_1\rho_1 + \sigma_2\rho_2} \leqslant 1. \qquad \ldots \text{(B)}.$$

Combining (A) and (B) we get the result. (See [2])

Theorem 2.2 : Let $\rho >> \bar{o}$ be an order point of f and let $\tau(>o)$ be the corresponding type of the function, then

$$\text{(i)} \quad \tau = \overline{\lim_{(\sigma_1, \sigma_1)\to\infty}} \; \overline{\lim_{(m, n)\to\infty}} \frac{\frac{\chi}{\rho}\left\{ | a_{mn} | \, e^{\sigma_1\lambda_m+\sigma_2\mu_n}\right\}^{\frac{1}{\chi | \rho}}}{e \,.\, e^{\sigma_1\rho_1+\sigma_2\rho_2}}$$

$$\text{(ii)} \quad \tau \leqslant \limsup_{(m, n)\to\infty} \left[\frac{\frac{\chi}{\rho} | a_{mn} |^{\frac{1}{\chi | \rho}}}{e}\right]$$

where $\dfrac{\chi(m, n)}{\rho} = \dfrac{\chi}{\rho} = \max.\left(\dfrac{\lambda_m}{\rho_1}, \dfrac{\mu_n}{\rho_2}\right)$.

Proof : Let $T > \tau$ (>0). Then there exists a $\sigma \epsilon R^2_+$ such that

$$\log M(\sigma) \leqslant T \exp(\sigma_1\rho_1+\sigma_2\rho_2) \text{ for } \sigma \geqslant \sigma^0 \qquad \ldots \text{(A)}$$

By Cauchy's inequality

$$| a_{mn} | \leqslant \frac{M(\sigma)}{\exp(\sigma_1\lambda_m+\sigma_2\mu_n)} \text{ for all } \sigma \text{ and } (m, n) \,\epsilon N^2.$$

$$\therefore \quad | a_{mn} | \leqslant \inf_{\sigma \geqslant \sigma} \,.\, \frac{\exp(T\, e^{\sigma_1\rho_1+\sigma_2\rho_2})}{\exp(\sigma_1\lambda_m+\sigma_2\mu_n)} \text{ for } (m, n) \,\epsilon N^2 \,.$$

which by ([4] Th. 3.4) is equivalent to the fact that

$$| a_{mn} | \leqslant \frac{\left[\dfrac{e.\, T.\, e^{\sigma_1^0\rho_1 + \sigma_2^0\rho_2}}{\chi/\rho}\right]^{\chi/\rho}}{e^{\sigma_1^0\lambda_m + \sigma_2^0\mu_n}}$$

for $(m, n) \in N^2 - J$

$$\leqslant \frac{\left[\frac{e.\, T.\, e^{\sigma_1 \rho_1 + \sigma_2 \rho_2}}{\chi/\rho}\right]^{\chi/\rho}}{e^{\sigma_1 \lambda_m + \sigma_2 \mu_n}}$$

for $(\sigma_1, \sigma_2) \geqslant (\sigma_1^0, \rho_2^0)$ and $(m, n) \in N^2 - J$.

$$\therefore \quad \frac{\chi}{\rho}\left[\,|\,a_{mn}\,|\, e^{\sigma_1 \lambda_m + \sigma_2 \mu_n}\right]^{\frac{1}{\chi/\rho}} < e\, T\, e^{\sigma_1 \rho_1 + \sigma_2 \rho_2}$$

for $(\sigma_1, \sigma_2) \geqslant (\sigma_1^0, \sigma_2^0)$ and $(m, n) \in N^2 - J$.

$$\therefore \quad \frac{\frac{\chi}{\rho}\left[\,|\,a_{mn}\,|\, e^{\sigma_1 \lambda_m + \sigma_2 \mu_n}\right]^{\frac{1}{\chi/\rho}}}{e.\, e^{\sigma_1 \rho_1 + \sigma_2 \rho_2}} \leqslant T$$

for $\sigma \geqslant \sigma^0$ and $(m, n) \in N^2 - J$.

Since τ is the infimum of the values of T for which (A) is satisfied

$$\therefore \quad \tau = \limsup_{(\sigma_1, \sigma_2) \to \infty}\ \limsup_{(m, n) \to \infty} \frac{\frac{\chi}{\rho}\left[\,|\,a_{mn}\,|\, e^{\sigma_1 \lambda_m + \sigma_2 \mu_n}\right]^{\frac{1}{\chi/\rho}}}{e.\, e^{\sigma_1 \rho_1 + \sigma_2 \rho_2}}$$

so (i) is proved.

$$\text{Again, since } \frac{(e^{\sigma_1 \lambda_m + \sigma_2 \mu_n})^{\frac{1}{\chi/\rho}}}{e^{\sigma_1 \rho_1 + \sigma_2 \rho_2}} = \frac{e^{\sigma_1 \rho_1}(e^{\sigma_2 \mu_n})^{\frac{1}{\lambda_m/\rho_1}}}{e^{\sigma_1 \rho_1 + \sigma_2 \rho_2}} \quad \text{if } \frac{\chi}{\rho} = \frac{\lambda_m}{\rho_1}$$

$$= e^{\sigma_2 \rho_1 \frac{\mu_n}{\lambda_m} - \sigma_2 \rho_2} = e^{\sigma_2 \left(\rho_1 \frac{\mu_n}{\lambda_m} - \rho_2\right)}$$

$$= e^{\frac{\sigma_2 \rho_1 \rho_2}{\lambda_m}\left(\frac{\mu_n}{\rho_2} - \frac{\lambda_m}{\rho_1}\right)}$$

$$\text{Since } \frac{\mu_n}{\rho_2} - \frac{\lambda_m}{\rho_1} \leqslant 0$$

$$\therefore \quad e^{\frac{\sigma_2 \rho_1 \rho_2}{\lambda_m}\left(\frac{\mu_n}{\rho_2} - \frac{\lambda_m}{\rho_1}\right)} \leqslant 1$$

$$\text{Similarly} \quad \frac{\left(e^{\sigma_1\lambda_m+\sigma_2\mu_n}\right)^{\frac{1}{\chi/\rho}}}{e^{\sigma_1\rho_1+\sigma_2\rho_2}} \leqslant 1 \quad \text{if } \frac{\chi}{\rho}=\frac{\mu_n}{\rho_2}$$

$$\therefore \quad \tau < \limsup_{(m,n)\to\infty} \frac{\frac{\chi}{\rho}\,|\,a_{mn}\,|^{\frac{1}{\chi/\rho}}}{e}.$$

Construction of entire functions with given integral order point and with a given type.

Theorem 2.3 : Let $(\alpha_1, \alpha_2) \in N^2$ and $a \in R_+$ be given. Then there exists an entire function f represented by a double Dirichlet series such that (α_1, α_2) is an order point and 'a' is the corresponding type of f.

Proof : Let us consider a double Dirichlet Series

$$f(s_1, s_2)=\sum_{t=1}^{\infty}\left(\frac{ea}{t}\right)^t \exp\,(ms_1+ns_2),\; m=t\alpha_1,\; n=t\alpha_2$$

$$\text{Since } a_{mn}=\left(\frac{ea}{t}\right)^t \text{ when } m=t\alpha_1,\; n=t\alpha_2$$

$$=0 \text{ otherwise;}$$

f satisfies the condition (1.3) and hence it is entire.

Now (ρ_1, ρ_2) is an order point of f iff [2]

$$\limsup_{(m,n)\to\infty} \frac{\frac{\chi}{\rho}\log\frac{\chi}{\rho}}{-\log|\,a_{mn}\,|}=1$$

$$\text{i.e., iff } \limsup_{t\to\infty} \frac{\max\left(\frac{m}{\rho_1}, \frac{n}{\rho_2}\right)\log\max\left(\frac{m}{\rho_1}, \frac{n}{\rho_2}\right)}{t\log t-t\log(ea)}=1$$

$$\text{i.e., iff } \limsup_{t\to\infty} \frac{t\max\left(\frac{\alpha_1}{\rho_1}, \frac{\alpha_2}{\rho_2}\right)\log\left\{t\max\left(\frac{\alpha_1}{\rho_1}, \frac{\alpha_2}{\rho_2}\right)\right\}}{t[\log t-\log(ea)]}=1$$

$$\text{i.e., iff } \limsup_{t\to\infty} \frac{\max\left(\frac{\alpha_1}{\rho_1}, \frac{\alpha_2}{\rho_2}\right)\left[\log t+\log\max\left(\frac{\alpha_1}{\rho_1}, \frac{\alpha_2}{\rho_2}\right)\right]}{\log t\left[1-\frac{\log(ea)}{\log t}\right]}=1$$

$$\text{i.e., iff } \max\left(\frac{\alpha_1}{\rho_1}, \frac{\alpha_2}{\rho_2}\right)=1.$$

For given $(\alpha_1, \alpha_2) \in N^2$ there exists an entire double Dirichlet series whose order pts. (ρ_1, ρ_2) satisfy the condition : $\max\left(\frac{\alpha_1}{\rho_1}, \frac{\alpha_2}{\rho_2}\right)=1$. Evidently (α_1, α_2) is an order point of f.

Next let us calculate the type of f corresponding to the order point (α_1, α_2). We know that corresponding to the order point (ρ_1, ρ_2) the type τ is given by

$$\tau = \limsup_{(\sigma_1, \sigma_2)\to\infty} \limsup_{(m, n)\to\infty} \frac{\frac{\chi}{\rho}\left[\,|\,a_{mn}\,|\, e^{\sigma_1\lambda_m+\sigma_2\mu_n}\right]^{\frac{1}{\chi/\rho}}}{e.\, e^{\sigma_1\rho_1+\sigma_2\rho_2}}$$

Now $\frac{\chi}{\rho}=\max\left(\frac{\lambda_m}{\rho_1}, \frac{\mu_n}{\rho_2}\right)=\max\left(\frac{t\alpha_1}{\alpha_1}, \frac{t\alpha_2}{\alpha_2}\right)=t.$

$$\therefore\quad \tau = \limsup_{(\sigma_1, \sigma_2)\to\infty} \limsup_{t\to\infty} \frac{t\left[\left(\frac{ea}{t}\right)^t \exp\left(\sigma_1 t\alpha_1+\sigma_2 t\alpha_2\right)\right]^{\frac{1}{t}}}{e.\exp\left(\sigma_1\alpha_1+\sigma_2\alpha_2\right)}$$

$$= \limsup_{(\sigma_1, \sigma_2)\to\infty} \limsup_{t\to\infty} \frac{t\left[\frac{ea}{t}.\exp\left(\sigma_1\alpha_1+\sigma_2\alpha_2\right)\right]}{e.\exp\left(\sigma_1\alpha_1+\sigma_2\alpha_2\right)}=a.$$

so the theorem is proved.

Definition : The set $\left[(\alpha_1, \alpha_2) : (\alpha_1, \alpha_2) \in R^2_+, \left(\frac{1}{\alpha_1}, \frac{1}{\alpha_2}\right) \in \rho_f\right.$ is called the reciprocal order set of $f \in F$ and is denoted by ρ_f^{-1}

Theorem 2.4 : The reciprocal order set ρ_f^{-1} of some $f \in F$ is convex.

Proof : Since log M (σ) is a convev function of σ [2] so for any $t \in R^2_+$ and $s \in R^2_+$ and any $\lambda \in [0, 1]$

$$\log M(\lambda t_1+\mu s_1, \lambda t_2+\mu s_2) \leqslant \lambda \log M(t_1, t_2)+\mu \log M(s_1, s_2)$$

where $\mu=1-\lambda$.

Now let us take two arbitrary points a and $b \in \rho_f$.

Now let us set,

$$t_i = \frac{\sigma_i a_i}{\lambda a_i + \mu b_i} \quad \text{and} \quad s_i = \frac{\sigma_i b_i}{\lambda a_i + \mu b_i},\ i = 1, 2$$

Then $\lambda t_1 + \mu s_1 = \dfrac{\lambda\sigma_1 a_1}{\lambda a_1 + \mu b_1} + \dfrac{\mu\sigma_1 b_1}{\lambda a_1 + \mu b_1} = \sigma_1$

Similarly, $\lambda t_2 + \mu s_2 = \sigma_2$

Since $a, b \in \rho_f$, $\log M(t_1, t_2) \leqslant \exp(t_1 b_1 + t_2 b_2)$ for $t > t_0$

and $\log M(s_1, s_2) \leqslant \exp(s_1 a_1 + s_2 a_2)$ for $s > s_0$

$$\therefore \log M(\sigma_1, \sigma_2) \leqslant \lambda \log M(t_1, t_2) + \mu \log M(s_1, s_2)$$
$$\leqslant \lambda \exp(t_1 b_1 + t_2 b_2) + \mu \exp(s_1 a_1 + s_2 a_2)$$
$$\text{for } t > t_0, \quad s > s_0$$

$$= \lambda \exp\left(\frac{\sigma_1 a_1 b_1}{\lambda a_1 + \mu b_1} + \frac{\sigma_2 a_2 b_2}{\lambda a_2 + \mu b_2}\right) + \mu \exp\left(\frac{\sigma_1 a_1 b_1}{\lambda a_1 + \mu b_1} + \frac{\sigma_2 a_2 b_2}{\lambda a_2 + \mu b_2}\right)$$
$$\text{for } \sigma > \sigma^0$$

$$= \exp\left(\frac{\sigma_1}{\frac{\lambda}{b_1} + \frac{\mu}{a_1}} + \frac{\sigma_2}{\frac{\lambda}{b_2} + \frac{\mu}{a_2}}\right) \text{ for } \sigma > \sigma^0$$

Consequently for any λ and μ, $\lambda \geqslant 0$, $\mu \geqslant 0$, $\lambda + \mu = 1$ the point

$\left(\dfrac{\lambda}{b_1} + \dfrac{\mu}{a_1}, \dfrac{\lambda}{b_2} + \dfrac{\mu}{a_2}\right)$ i.e. any point of the segment joining the pts. (b_1^{-1}, b_2^{-1}) and (a_1^{-1}, a_2^{-1}) of ρ_f^{-1} is also in ρ_f^{-1}. Thus ρ_f^{-1} is convex.

The above theorem is equivalent to the following:

The set $\rho'_f \subset \rho_f$ where $\rho'_f = \{\alpha \,;\, \alpha \in R^2_+, \alpha \in \rho_f)$ is reciprocally convex.

Theorem 2.5: The set ρ'_f of some $f \in F$ is convex,

Let $a = (a_1, a_2)$, $b = (b_1, b_2) \in \rho'_f$ and let $C = pa + qb$ where $p \geqslant 0$, $q \geqslant 0$, $p + q = 1$. Since the set ρ'_f is reciprocally convex it follows that $d = (d_1, d_2) \in \rho'_f$ where $\dfrac{1}{d_j} = \dfrac{p}{a_j} + \dfrac{q}{b_j}$, $j = 1, 2$.

Now for $j = 1, 2$

$$(c_j - d_j)(pb_j + qa_j) = (p^2 + q^2)(a_j b_j) + pq(a_j^2 + b_j^2) - a_j b_j$$
$$= (p + q)^2 a_j b_j + pq(a_j - b_j)^2 - a_j b_j$$
$$\geqslant (p + q)^2 a_j b_j - a_j b_j = 0$$

which shows that $c \geqslant d$ and hence $c \in \rho'_f$.

This implies that the set ρ'_f is convex.

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