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ON BIGROUPOIDS

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The purpose of this note is to characterise a certain classes of monoids in terms of the corresponding classes of bigroupoids.

A monoid is a semigroup with identity. An involution of a monoid is a unary operation $t : S \rightarrow S$ satisfying the following axioms

(1) $(x^t)^t = x$ (2) $(xy)^t = y^t x^t$

A monoid is called a monoid with involution, if we assign an involution to it.

An algebraic system (B, $*$, o) of type (2, 2) which is taken into consideration here, always satisfies the following axiom B. Let us call this system bigroupoid.

(B) $x \circ (y * z) = (z \circ x) * y$.

1. Bigroupoid with identity.

Definition. An element u of a bigroupoid S is called a left (right) identity of S, if it satisfies

$(u_L) : u * a = a$ $(u_R : a \circ u = a)$

Lemma 1. Let u be a left identity and v be a right identity of a bigroupoid S. Then (i) $v * u = v$ and (ii) $v \circ u = u$.

Proof : (i) $v * u = (v \circ v) * u$
$= v \circ (u * v)$ (by B)
$= v \circ v$
$= v.$

(ii) $v \circ u = v \circ (u * u)$
$= (u \circ v) * u$
$= u * u$
$= u.$

Lemma 2. Let u be a left identity and v be a right identity then $a * u = v \circ a$ for all $a \in$ S.

Proof : $a * u = (a \circ v) * u$
$= v \circ (u * a)$
$= v \circ a.$

Proposition 1. If S contains a left identity u then there exists atmost one right identity.

Proof : Let v and v_1 be two right identies. Then

$$\begin{aligned} v_1 &= u * v_1 && \text{(by } u_L) \\ &= (u \circ v) * v_1 && \text{(by } u_R) \\ &= v \circ (v_1 * u) && \text{(by B)} \\ &= v \circ v_1 && \text{(by Lemma 1)} \\ &= v. && \text{(by } u_R) \end{aligned}$$

Proposition 2. If S contains a right identity v then there exists atmost one left identity in S.

Proof : Let u and u_1 be two left identities of S.

$$\begin{aligned} u_1 &= u_1 \circ v && \text{(by } u_R) \\ &= u_1 \circ (u * v) && \text{(by } u_L) \\ &= (v \circ u_1) * u && \text{(by B)} \\ &= u_1 * u && \text{(by Lemma 1)} \\ &= u. && \text{(by } u_L) \end{aligned}$$

Definition. A pair (u, v) is called an identity element of S if u is a left identity and v is a right identity.

Proposition 3. S contains atmost one identity (u, v).

Proof : This follows from Proposition 1 and Proposition 2.

Definition. $J_u(x) = x * u$ and $J_v(x) = v \circ x$.

Lemma 3. $J_u = J_v$,

Proof : Let $x \in$ S. From Lemma 2,

$J_u(x) = x * u = v \circ x = J_v(x)$.

Hence $J_u = J_v$.

Lemma 4. (a) $J_u(x * y) = y * x$ (b) $J_u(x \circ y) = y \circ x$.

Proof :

$$\begin{aligned} J_u(x * y) = J_v(x * y) &= v \circ (x * y) \\ &= (y \circ v) * x && \text{(by B)} \\ &= y * x. && \text{(by } u_R) \end{aligned}$$

$$\begin{aligned} J_u(x \circ y) &= (x \circ y) * u \\ &= y \circ (u * x) && \text{(by B)} \\ &= y \circ x. && \text{(by } u_L) \end{aligned}$$

Lemma 5. $J_u(J_u(x))=x$.

Proof : $J_u(J_u(x))=J_u(J_v(x))$

$=(v \circ x) * u$

$=x \circ (u * v)$ (by B)

$=x \circ v$ (by u_L)

$=x$.

Lemma 6. Let S be a bigroupoid with the identity (u, v). Then $(a * b) \circ c=a * (b \circ c)$

Proof : Let a, b, c be three elements of S. Then

$(a * b) \circ c=J_u(c \circ (a * b))$ (by Lmma 4 (b))

$=J_u((b \circ c) * a)$ (by B)

$=a * (b \circ c)$ (by Lemma 4(a)).

Theorem 1. Let S be a bigroupoid with the identity (u, v). If we define $a.b=J_u(a) * b$, then (S, J_u) is a monoid with involution, where u is the identity of this monoid.

Proof : Let a, b, c be three elements of S. Then

$a . (b . c)=a . (J_u(b) * c)$

$=J_u(a) * (J_u(b) * c)$

$=J_u(a) * (J_v(b) * c)$ (by Lemma 3)

$=J_u(a) * ((v \circ b) * c)$

$=J_u(a) * (b \circ (c * v))$ (by axiom B)

$=(J_u(a) * b) \circ (c * v)$ (by Lemma 6)

$=(v \circ (J_u(a) * b)) * c$ (by axiom B)

$=((b \circ v) * J_u(a)) * c$

$=(b * J_u(a)) * c$

$=J_u(J_u(a) * b) * c$ (by Lemma 4)

$=(a . b) . c$.

Hence associative property holds in (S, .).

Now for any $a \in S$, we have

$u . a = J_u(u) * a = (u * u) * a = u * a = a$, and

$a . u = J_u(a) * u = (a * u) * u = a$ (by Lemma 5)

Hence u is the identity element of the semigroup (S, .). Let us now show that J_u is an involution in (S, .). Let $x \in S$. Then $J_u(J_u(x)) = x$ (by Lemma 5) and $J_u(a \, . \, b) = J_u(J_u(a) * b) = b * J_u(a) = J_u(J_u(b)) * J_u(a) = J_u(b) \, . \, J_u(a)$. Hence $J_u : S \to S$ is an involution in (S, .). Hence the theorem.

We can also prove the following theorem :

Theorem 2. Let (u, v) be the identity of the bigroupoid S. If we define $a \, . \, b = a \circ J_v(b)$ then $(S, ., J_v)$ is a monoid with involution where v is the identity of this monoid.

Let us denote by S_u the monoid obtained in Theorem 1 and S_v the monoid obtained in Theorem 2.

Theorem 3. S_u and S_v are isomorphic to each other.

Proof : Let us define a mapping f from S_v to S_u by $f(u) = a \circ u$ for $a \in S_v$. Then,

$$
\begin{aligned}
f(a \, . \, b) &= (a \, . \, b) \circ u \\
&= (a \circ J_v(b)) \circ u && \text{(by the definition of } a \, . \, b \text{ in } S_v) \\
&= (a \circ J_u(b)) \circ u && \text{(by Lemma 3)} \\
&= (a \circ (b * u)) \circ u \\
&= ((u \circ a) * b) \circ u && \text{(by axiom B)} \\
&= (u \circ a) * (b \circ u) && \text{(by Lemma 6)} \\
&= J_u(a \circ u) * (b \circ u) && \text{(by Lemma 4(b))} \\
&= (a \circ u) \, . \, (b \circ u) && \text{(by the definition of } a \, . \, b \text{ in } S_u) \\
&= f(a) \, . \, f(b).
\end{aligned}
$$

Let $a, b \in S_v$ such that $f(a) = f(b)$. Then $a \circ u = b \circ u$. From this $J_u(a \circ u) = J_u(b \circ u)$. Then by Lemma 4(*b*), $u \circ a = u \circ b$. $(u \circ a) * v = (u \circ b) * v$.

This implies $a \circ (v * u) = b \circ (v * u)$. Then by Lemma 1, $a \circ v = b \circ v$. Hence $a = b$ if $f(a) = f(b)$. Let $a \in S$. Then $(v \circ a) * v \in S$.

Now $f((v \circ a) * v) = ((v \circ a) * v) \circ u = (v \circ a) * (v \circ u) = (v \circ a) * u$ (by Lemma 1) $= a \circ (u * v) = a \circ v = a$. Thus it follows that f is an isomorphism of S_v onto S_u.

Note. Let S be a monoid with an involution. If we define $a * b = a^t b$ and $a \circ b = ab^t$ where a^t denotes the involution of a, then we can show that S is a bigroupoid with identity.

2. Bigroupoid and Group.

In [(2), P 73] we have considered a bigroupoid S that satisfies the following axiom :

$$G : a \circ (b * a) = b \text{ for all } a, b \in S$$

Some results abount this system are listed below :

Lemma 7. [(2) P 72] $a * a = b * b$ for $a, b \in S$

Let $u = a * a = b * b = c * c = \cdots$ for $a, b, c \in S$.

Lemma 8. [(2), P 72] $u * d = d$ for all $d \in S$.

Lemma 9. [(2), P 73] $d \circ u = d$ for all $d \in S$.

Theorem 4. [(2), P 73]. Let S be a bigroupoid satisfying the axiom G. If we define $a \,.\, b = J_u(a) * b$ for all $a, b \in G$, then G is a group.

3. Regular bigroupoid.

Definition. An element a of a bigroupoid S is said to be regular if $a = (a \circ a) * a$. If every element of S is regular then S is said to be a regular bigroupoid.

Definition. An element $a \in S$ is said to be left (right) idempotent if $a * a = a$ $(a \circ a = a)$.

Leema 10. In a regular bigroupoid every left idempotent is a right idempotent and conversely.

Proof : Let a be a left idempotent then $a * a = a$. Now $a \circ a = a \circ (a * a) = a$. Conversely, assume that a is right idempotent.

Then $a \circ a = a$. Hence $a * a = (a \circ a) * a = a \circ (a * a) = a$.

Lemma 11. In a regular bigroupoid $a \circ a$ and $a * a$ are idempotents.

Proof : We have $a \circ (a * a) = a$. Then $(a \circ a) * a = a$. From this, $a \circ ((a \circ a) * a) = a \circ a$. This implies $(a \circ a) * (a \circ a) = a \circ a$. Hence $a \circ a$ is a left idempotent. From Lemma 10, it follows that $a \circ a$ is idempotent.

Again $a \circ (a * a) = a$ we find that $(a \circ (a * a)) * a = a * a$.

This implies $(a * a) \circ (a * a) = a * a$. Hence $a * a$ is a right idemptent. Then from Lemma 10, it follows that $a * a$ is an idempotent.

Theorem 5. A bigroupoid is a group if and only if it is regular and contains only one idempotent.

Proof. Suppose that the bigroupoid S is regular and contains only one idempotent. Then from leema 11, it follows that $a \circ a = a * a = b \circ b = b * b$ for any $a, b \in$ S. Hence $a \circ (b * a) = (a \circ a) * b = (b \circ b) * b = b \circ (b * b) = b$. This shows that S satisfies the axiom G. Then from Theorem 4 we find that S is a group. Conversely, suppose that S is a group. Then $a \circ (b * a) = b$ for any $b \in$ S. Hence $a \circ (a * a) = a$. This implies that S is regular. Then from Lemma 8 and Lemma 9 we have $a * a$ is the identity of S. Let e be an idempotent in S. Hence $e = e \circ e = e * e$. This shows that e is the identity of S. Hence S contains only one idempotent.

A semigroup S with an involution t is called [(1), P 370] a t- regular semigroup if it satisfies the axiom

$$x = x\, x^t\, x$$

If we define $a * b = a^t\, b$ and $a \circ b = ab^t$, then (S, * o) is a regular bigroupoid.

Definition. A bigroupoid S is said to be commutative if $a * b = b \circ a$, for any $a, b \in$ B.

Theorem 6. A commutative regular bigroupoid is a disjoint union of groups.

Proof : Let E be the set of all distinct idempotents of a commutative regular bigroupoid S. Suppose $e \in$ E. Let $G_e = \{a \in S : e * a = a$ and there exists $a' \in$ S with the properties (i) $a' * a = e$ (ii) $e * a' = a\}$.

Since S is commutative, we have $e * a = a \circ e = a$ for all $a \in G_e$.

Suppose $a, b \in G_e$. Then $e * (a * b) = (e \circ e) * (a * b) = e \circ ((a * b) * e)$
$= e \circ ((b \circ a) * a) = e \circ (a \circ (e * b)) = e \circ (a \circ b) = e \circ (b * a) = (a \circ e) * b$
$= (e * a) * b = a * b$. Since $a, b \in G_e$, there exist $a', b' \in$ S, such that $a' * a = e$ and $b' * b = e$.

Now $(a' * b') * (a * b) = (b' \circ a') * (a * b) = a' \circ ((a * b) * b')$
$= a' \circ ((b \circ a) * b') = a' \circ (a \circ (b' * b)) = a' \circ (a \circ e) = a' \circ (e * a)$
$= a' \circ a = a * a' = (a \circ e) * a' = e \circ (a' * a) = e \circ e = e$.

Also $e * (a' * b') = (e \circ e) * (a' * b') = e \circ ((a' * b') * e)$
$= e \circ (e \circ (a' * b')) = e \circ ((b' \circ e) * a' = e \circ (b' * a') = (a' \circ e) * b'$
$= a' * b'$. Hence $a * b \in G_e$ for all $a, b \in G_e$. Since $a \circ b = b * a$, it follows that $a \circ b \in G_e$ for all $a, b \in G_e$. Hence G_e is a bigroupoid. Let f be an idempotent of S such that $f \in G_e$. Then $e * f = f$ and $g * f = e$ for some $g \in$ S. Also $e * g = g$. Now $f = e * f = f \circ e = f \circ (g * f) = (f \circ f) * g = f * g = (f \circ e) * g$

$=e \circ (g * f)=e \circ e=e$. Hence G_e contains only one idempotent. Also it is true that G_e is a regular bigroupoid. Then from Theorem 5, it follows that G_e is a group. Let $a \in S$. Then $(a \circ a) * a=a$ implies that $e=a \circ a= a \circ ((a \circ a) * a)$ $=(a \circ a)$ is an idempotent and $e * a=a$. Also $e=a \circ a=a * a$. Hence $a \in G_e$. Let e, f be two dstinct idempotents of S. Suppose $a \in G_e \cap G_f$. Then there exist t and t_1 such that $t * a=e$, $e * t=t$, $t_1 * a=f$ and $f * t_1=t_1$. Now from $t * a=e$ we have $f \circ (t * a)=f \circ e$. Hence $f \circ e=(a \circ f) * t=a * t$ (since $a \in G_f$) $=(a \circ e) * t$ (since $a \in G_e$) $=e \circ (t * a)=e \circ e=e$. Then $e=f \circ e=e * f=$ $(f \circ e) * f=e \circ (f * f)=e \circ f=e \circ (t_1 * a)=(a \circ e) * t_1=a * t_1=(a \circ f) * t_1$ $=f \circ (t_1 * a)=f \circ f=f$. This is a contradiction. Hence $G_e \cap G_f=\phi$ when $e \neq f$.

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