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COMMON FIXED POINT THEOREM IN 2-METRIC SPACES

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Introduction : S. Gahler [2] introduced the notion of 2-metric space as follows :

A set X is defined to be 2-metric space, if there exists a real valued function $d : X \times X \times X \rightarrow R^+$ satisfying the following conditions :

(i) to each pair of points x, y $(x \neq y)$ of X there is one $z \in X$ such that $d(x, y, z) \neq 0$

(ii) $d(x, y, z) = 0$ only when at least two of three points are equal

(iii) $d(x, y, z) = d(x, z, y) = d(y, z, x) = \cdots$

(iv) $d(x, y, z) \leqslant d(x, y, u) + d(x, u, z) + d(u, y, z)$.

In the work of I. Keyoshi [3] we find the following definitions.

Definition 1 : If $d(x, y, z)$ is bounded, the 2-metric space is said to be bounded. By its diameter we mean $\sup\limits_{x, y, z \in X} d(x, y, z)$

Definition 2 : If $d(x_n, x, a)$ converges to zero for all $a \in X$ we say that the sequence $\{x_n\}$ converges to x and x is a limit of $\{x_n\}$.

Definition 3 : If in a 2-metric space X, $d(x_m, x_n, a) \rightarrow 0$ $(m, n \rightarrow \infty)$ for all $a \in X$, the sequence $\{x_n\}$ is called a Cauchy sequence.

If in X, every Cauchy sequence is convergent, X is called complete.

Also in the 2-metric space, the notion of continuity of a self-mapping may be given as follows :

Definition 4 : If in a 2-metric space X, $d(x, x_0, a) \rightarrow 0$ implies $d(Tx, Tx_0, a) \rightarrow 0$ for all $a \in X$, then we say that $T : X \rightarrow X$ is continuous at $x = x_0$.

Here we shall prove a fixed point theorem which is proved for a 1-metric space by B. Fisher [1].

Theorem : Suppose S and T are continuous mappings of the complete and bounded 2-metric space X into itself. If S and T satisfy the condition

$$d(S^2x, T^2y, a) \leqslant C \max [d(x, Ty, a), d(y, Sx, a), d(x, y, a)]$$

for all x, y, a in X, where $0 \leqslant C < 1$. Then S and T have a unique common fixed point u.

Proof : Let x be any arbitrary point in X. Then

$$d(S^n x, T^r x, a)$$
$$\leqslant C \max [d(S^{n-2}x, T^{r-1}x, a), d(S^{n-1}x, T^{r-2}x, a), d(S^{n-2}x, T^{r-2}x, a)]$$
$$\leqslant C^2 \max [d(S^{n-4} x, T^{r-2} x, a), d(S^{n-3} x, T^{r-3} x, a), d(S^{n-4} x, T^{r-3} x, a), d(S^{n-2} x, T^{r-4} x, a), d(S^{n-3} x, T^{r-4} x, a), d(S^{n-4} x, T^{r-4} x, a)]$$

and so on, till powers of S and T do not become negative.

Since X is bounded,

$$M = \text{Sup} [d(x, y, a), x, y, a \in X] < \infty.$$

For arbitary $\epsilon > 0$, choose N so that $C^N M < \epsilon/3$

$\therefore$ $d(S^n x, T^r x, a) < \epsilon/3$ for $n, r \geqslant 2N$

and so $d(S^n x, S^m x, a) \leqslant d(S^n x, T^r x, a) + d(T^r x, S^m x, a) + d(S^n x, S^m x, T^r x)$

$$< 2\epsilon/3 + d(S^n x, T^r x, S^m x)$$
$$< 2\epsilon/3 + C^N M$$
$$< 2\epsilon/3 + \epsilon/3 = \epsilon$$

for $m, n, r \geqslant 2N$.

Hence $\{S^n x\}$ is a Cauchy sequence in the complete 2-metric space X and so has a limit u in X. Since S is continuous $Su = u$ and so u is a fixed point of S.

Similarly $\{T^n x\}$ is a Cauchy sequence in X and since $d(S^n x, T^n x, a) < \epsilon/3$ for $n \geqslant 2N$, the sequence $\{T^n x\}$ also converges to u.

As T is continuous, $Tu = u$ and so u is a common fixed point of S and T.

If possible, let w be another common fixed point of S and T. Then

$$d(u, w, a) = d(S^2 u, T^2 w, a)$$
$$\leqslant C \max [d(u, Tw, a), d(Su, w, a), d(u, w, a)]$$
$$= C\, d(u, w, a).$$

As $C < 1$, $u = w$ and so the common fixed point is unique.

Corollary : Let S and T be continuous mappings of the complete and bounded 2-metric space X into itself satisfying the inequality

$$d(S^2 x, T^2 y, a) \leqslant C \max [d(x, Ty, a), d(y, Sx, a)] \text{ for all } x, y, a \in X,$$

where $0 \leqslant C < 1$.

Then S and T have a unique common fixed point.

Proof : Since $d(S^2x, T^2y, a) \leqslant C \max [d(x, Ty, a), d(y, Sx, a)]$
$\leqslant C \max [d(x, Ty, a), d(y, Sx, a), d(x, y, a)]$

for all x, y in X, the result follows from the theorem (above).

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