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ON THE DISTRIBUTION OF THE EIGENVALUES OF A DIFFERENTIAL SYSTEM

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Abstract : The object of the present paper is to investigate certain asymptotic relations connecting the infinite series expansions involving eigenvalues associated with the differential system $(-D^2+P)\,U=\lambda U$ and the integrals involving the characteristic roots of P, which is a positive definite symmetric 2×2 matrix. Tauberian theorems are next applied to obtain, interalia, the asymptotic distribution of $N(\lambda)$, the number of eigenvalues less than λ.

1. The Problem

Let the differential system be

$$M\,\phi = \lambda\phi, \qquad \dots (1.1)$$

where $M \equiv \begin{pmatrix} -D^2+p & r \\ r & -D^2+q \end{pmatrix}$, $D \equiv \dfrac{d}{dx}$

p, q, r are real valued functions of x, with derivatives, which are alsolutely continuous over any compact sub-interval of $R : [0, \infty)$ and λ is a complex parameter.

The boundary conditions considered are

$$u(0) = v(0) = 0 \qquad \dots (1.2)$$

Or

$$u'(0) = v'(0) = 0 \qquad \dots (1.3)$$

where $\phi \equiv \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$.

The Hilbert space H in which the theory associated with the operator M, is developed, is that of functions $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ for which $\int_0^\infty (|f_1|^2 + |f_2|^2)\,dx < \infty$, with usual definition of the inner product.

It is well known that the differential system (1.1) along with the boundary conditions (1.2) (or with (1.3)) gives rise to the Dirichlet (or the Neumann) eigenvalue problems. It is assumed that $\begin{pmatrix} u \\ v \end{pmatrix} \in L_2$ at infinity.

Let us further assume that $p\phi$, $q\phi$, $r\phi \in H$ and that $p, q > 0$, det $P \geqq 0$, in $[0, \infty)$, $P \equiv \begin{pmatrix} p & r \\ r & q \end{pmatrix}$. Also let P be pseudo-monotonic over $[0, \infty)$ in the sense that for $j \geqq k$, $p_j \geqq p_k$, $q_j \geqq q_k$, det $(P_j - P_k) \geqq 0$, (p_i, q_i, P_i are the p, q, P, in which x is replaced by x_i), $j, k = 0, 1, 2, \ldots$ Then the sequence of eigen values $\{\lambda_n\}$ is positive, and the spectrum is discrete with $\lim_{n \to \infty} \lambda_n = \infty$, over $[0, \infty)$, both for the Dirichlet and the Neumann eigenvalue problems (see Chakravarty and Sengupta [2]).

In particular $\lambda_n \geqq \lambda_0 \geqq 0$, λ_0 being the least eigenvalue of the system.

Let $\psi_n(x) \equiv \begin{pmatrix} \psi_{1n} \\ \psi_{2n} \end{pmatrix}$ be the eigenvector corresponding to the eigenvalue λ_n, and let $\Delta \equiv \Delta(x) \equiv \frac{1}{2}\left(p+q+\sqrt{(p-q)^2+4r^2}\right)$,

$\eta \equiv \eta(x) \equiv \frac{1}{2}\left(p+q-\sqrt{(p-q)^2+4r^2}\right)$ be the characteristic roots of the matrix P.

Then Δ, η are both real and both steadily increase with x, if $p\ (> q)$, r are steadily increasing and $(p-q)q' - 2rr' \geqq 0$, is satisfied (see Chakravarty and Sengupta [2]). We take into account the following conditions :

(a) $|p(\xi)-p(x)|$, $|q(\xi)-q(x)|$, $|r(\xi)-r(x)| < C|\xi-x|\,\eta^{a_0}(x)$ for

$0 < |\xi - x| \leqq 1$, C, a_0 are positive constants, $0 < a_0 < \frac{5}{4}$.

(b) $p(\xi)$, $q(\xi)$, $|r(\xi)| \leqq K_0 \exp\left\{\frac{1}{2}|\xi-x|\,\eta^{a_1}(x)\right\}$ for $|\xi-x| > 1$;

K_0, a_1 are positive constants, $0 < a_1 < \frac{1}{2}$.

(c) $\eta(x) \geqq x^2\,\eta^{2a_1}(b)$ for large $x \in R = (0, \infty)$, $b > 0$.

(d) $\Delta(x)$ as well as $\eta(x)$ are steadily increasing with x.

Let there exist two well-behaved functions $t_j(x)$, $j = 1, 2$ on R, such that

(e) $t_j(\xi) < C_j t_j(x)$, $|\xi - x| \leqq 1$; $1 < t_j \leqq \eta \leqq \Delta$, for large x, $j = 1, 2$, where C_j are positive constants.

(f) $\int_0^\infty \frac{dx}{t_j^{A_j}} < \infty$, $t_j \equiv t_j(x)$, $j = 1, 2$, for some positive numbers A_1, A_2.

(g) $t_1^{2s}\psi_{1n}^2 + t_{2n}^{2s}\psi_{2n}^2 \in L[0, \infty)$, where s is a positive integer $\geqq 2$.

We define $a_n^{(\tau_1, \tau_2, m)} = \int_0^\infty \left(t^\tau(x), \psi_n^2(x)\right) \Delta^{-2m}(x)\, dx$ ... (1.4)

and $b_n^{(\tau_1, \tau_2, m)} = \int_0^\infty \left(t^\tau(x), \psi_m^2(x)\right) \eta^{-2m}(x)\, dx$ ... (1.5)

where $(t^\tau, \psi_n^2) = \sum_{j=1}^{2} t_j^{\tau_j} \psi_{jn}^2$, m and τ_j are positive numbers,

satisfying

(h) : $0 \leqq \tau_j \leqq \min\{2m - A_j, 2s - A_j - 5/2\}$, $j=1, 2$

and when $m = 0$,

(h′) : $0 \leqq \tau_j \leqq 2s - A_j - 1$, $j=1, 2$.

Put $S_{\tau_1, \tau_2, s, m} = \sum_{n=0}^{\infty} a_n^{(\tau_1, \tau_2, m)} (\lambda_n + \mu)^{-2s-2m}$, $\mu \geqq 1$,

$$I_{\tau_1, \tau_2, s, m} = \frac{1.3.5 \ldots\ldots (4s-3)}{2^{2s}(2s-1)!} \int_0^\infty \frac{t_1^{\tau_1}(x) + t_2^{\tau_2}(x)}{(\mu+\Delta)^{2(s+m)-1/2}}\, dx,$$

$\Delta \equiv \Delta(x)$, and $S^*_{\tau_1, \tau_2, s, m}$, $I^*_{\tau_1, \tau_2, s, m}$ are the corresponding entities, in which $\Delta \equiv \Delta(x)$ is replaced by $\eta \equiv \eta(x)$.

Our object in the present paper is to show first that as $\mu \to \infty$,

$$S_{\tau_1, \tau_2, s, m} \sim I_{\tau_1, \tau_2, s, m};\quad S^*_{\tau_1, \tau_2, s, m} \sim I^*_{\tau_1, \tau_2, s, m}.$$

Then by means of certain Tauberian theorems, we obtain the asymptotic estimates for $a_n^{(\tau_1, \tau_2, m)}$ and $b_n^{(\tau_1, \tau_2, m)}$, and from these for $N(\lambda)$ the number of eigenvalues not exceeding λ.

2. Some preliminary results

Let $G(\xi, y, \mu) \equiv \begin{pmatrix} G_{11}(\xi, y, \mu) & G_{21}(\xi, y, \mu) \\ G_{12}(\xi, y, \mu) & G_{22}(\xi, y, \mu) \end{pmatrix}$ be the Green's matrix for the system (1.1), with (1.2) or (1.3), where $\lambda = -\mu$, $\mu \geqq 1$, in the singular case $0 \leqq \xi < \infty$, the Green's vectors being $G_j = \begin{pmatrix} G_{j1} \\ G_{j2} \end{pmatrix}$, $j=1, 2$.

Also let $g(\xi, y, k)$ be the corresponding Green's matrix for the system

$$M_0 \phi = -k^2(x) \phi, \quad M_0 \equiv \begin{pmatrix} -D^2 & 0 \\ 0 & -D^2 \end{pmatrix}, \quad D \equiv d/d\xi, \ \phi \equiv \phi(\xi)$$

satisfying the same Dirichlet (or Neumann) boundary conditions as before and $0 \leqq \xi < \infty$ with vectors $g_j \equiv \begin{pmatrix} g_{j1} \\ g_{j2} \end{pmatrix}$, $j=1, 2$.

Then both G_j and $g_j \in L_2$ $(j=1, 2)$, G_j, g_j satisfy the same boundary conditions at $\xi=0$, and by a variant of the analysis adopted by Sengupta [5] [Lemma 2, P-101],

$\lim_{b \to \infty} \left[G_j(b, \xi, y, \mu), g_i(b, \xi, y, k) \right] = 0$, where $G_j(b, \cdots)$, $g_i(b, \cdots)$ are the Green's vectors for the interval $[0, b]$, such that $\lim_{b \to \infty} G_j(b, \cdots) = G_j(\,.\,)$ and $\lim_{b \to \infty} g_i(b, \cdots) = g_i(\,.\,)$ and $[U, V]$ represents the bilinear concomitant of the vectors U, V. (For definitions see Chakravarty [1], 1965).

Then by making use of the properties of the Green's matrix, it follows after some easy manipulations, that

$$G^T(x, y, \mu) = g^T(x, y, k) + \int_0^\infty G(y, \xi, \mu)\,(P(\xi) - P(x))\, g^T(x, \xi, k)\, d\xi$$

$$+ \int_0^\infty G(y, \xi, \mu)\, [\, P(x) - (k^2(x) - \mu)\, I\,]\, g^T(x, \xi, k)\, d\xi \qquad \ldots (2.1)$$

where I is the unit matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $P(\xi)$ is the matrix P with $x = \xi$, G^T, g^T being the transpose of G and g respectively.

Since $k \equiv k(x)$ is at our disposal, to simplify our discussions, we so choose it, that the last integral on the right hand side of (2.1) vanishes. We further assume that the two matrices $A \equiv g^T(x, \xi, k)$

and
$$B = \begin{pmatrix} p-(k^2-\mu) & r \\ r & q-(k^2-\mu) \end{pmatrix}$$

commute (this is implied in our discussion in view of the explicit expression for the elements of $g^T(x, \xi, k)$). We then have $k^2 = \mu + \Delta(x)$ or $k^2 = \mu + \eta(x)$, where $\Delta(x)$ and $\eta(x)$ are the characteristic roots of the matrix $P(x)$.

The equation (2.1) now takes the simpler form

$$G^T(x, y, \mu) = g^T(x, y, k) + \int_0^\infty G(y, \xi, \mu)\,(P(\xi) - P(x))\,g^T(x, \xi, k)\,d\xi \qquad \ldots (2.2)$$

where $k^2 = \mu + \Delta(x)$ or $\mu + \eta(x)$, $\mu \geqq 1$, under the assumptions made above. (2.2) is the basis for our investigations that follow.

It is easy to deduce that

$$\frac{\psi_n(x)}{\lambda_n + \mu} = -\int_0^\infty G(x, y, \mu)\,\psi_n(y)\,dy \qquad \ldots (2.3)$$

Therefore by the expansion formula

$$G_j(y, x, \mu) = -\sum_{n=0}^{\infty} \frac{\psi_{jn}(x)\psi_n(y)}{(\lambda_n + \mu)}, \quad (j=1, 2) \qquad \ldots (2.4)$$

Let $D_\mu^{(n)}$ be the symbol of differentiation n times with respect to μ.

Then from (2.4)

$$D_\mu^{(s-1)} G_j(y, x, \mu) = (-1)^s (s-1)! \sum_{n=0}^{\infty} \frac{\psi_{jn}(x)\,\psi_n(y)}{(\lambda_n + \mu)^s} \qquad \ldots (2.5)$$

from which by the Parseval relation,

$$\frac{1}{\{(s-1)!\}^2} \| D_\mu^{(s-1)} G_j^T(x, y, \mu) \|_{0,\infty} = \sum_{n=0}^{\infty} \frac{\psi_{jn}^2(x)}{(\lambda_n + \mu)^{2s}}, \quad (j=1, 2) \qquad (2.6)$$

where $\|\omega\|_{0,\infty} = \int_0^\infty (\omega_1^2 + \omega_2^2)\,dy$, $\omega \equiv \omega(y) \equiv \begin{pmatrix} \omega_1(y) \\ \omega_2(y) \end{pmatrix}$

and $G_j^T(\cdot) = \begin{pmatrix} G_{1j} \\ G_{2j} \end{pmatrix}$ is the Green's vector corresponding to $G^T(.)$, the transpose of the Green's matrix G (.).

In the second term of the right hand side of (2.2) substitute for the Green's vectors $G_j(y, \xi, \mu)$ by the relations (2.4) and then differentiate $(s-1)$ times with respect to μ, by using the Leibnitz formula. Then it follows from (2.2) that

$$\frac{1}{(s-1)!} D_\mu^{(s-1)} G_1^T(x, y, \mu) = (-1)^s \sum_{n=0}^{\infty} \frac{\psi_{1n}(x)\,\psi_n(y)}{(\lambda_n+\mu)^s}$$

$$= \frac{1}{(s-1)!} D_\mu^{(S-1)} g_1(y, x, k) + J_S(x, y, k) \qquad \ldots (2.7)$$

where $J_S(x, y, k) \equiv \begin{pmatrix} J_{1s}(x, y, k) \\ J_{2s}(x, y, k) \end{pmatrix}$

$$= \sum_{j=0}^{s-1} C_{js} \sum_{n=0}^{\infty} \left(\int_0^{\infty} \psi_n^T(\xi) \Big(P(\xi) - P(x) \Big) D_\mu^{(s-j-1)} g_1(\xi, x, k)\, d\xi \right) \frac{\psi_n(y)}{(\lambda_n+\mu)^{j+1}}$$

$$= \sum_{j=0}^{s-1} C_{js} \sum_{n=0}^{\infty} \chi_n(x) \frac{\psi_n(y)}{(\lambda_n+\mu)^{j+1}} \qquad \ldots (2.8)$$

C_{js} are the numerical constants (suitably adjusted) obtained in the process of application of the Leibnitz formula, and

$$\chi_n(x) = \int_0^{\infty} \psi_n^T(\xi) \Big(P(\xi) - P(x) \Big) D_\mu^{(s-j-1)} g_1(\xi, x, k)\, d\xi.$$

A similar result holds for $D_\mu^{(s-1)} G_2^T(x, y, \mu)$,

Since $g_{12}=0$, it follows from (2.6) and the relation (2.7) that

$$\sum_{n=0}^{\infty} \frac{\psi_{1n}^2(x)}{(\lambda_n+\mu)^{2s}} = \int_0^{\infty} \left[\frac{1}{(s-1)!} D_\mu^{(s-1)} g_{11}(y, x, k) + J_{1s}(x, y, k) \right]^2 dy$$

$$+ \int_0^{\infty} \Big[J_{2s}(x, y, k) \Big]^2 dy \qquad \ldots (2.8a)$$

$$\leqslant\left[\left(\frac{1}{\{(s-1)!\}^2}\int_0^\infty\left(D_\mu^{(s-1)}g_{11}(y,x,k)\right)^2dy\right)^{1/2}+\|J_s(x,y,k)\|_{0,\infty}^{1/2}\right]^2 \quad \dots (2.9)$$

by using the Minkowsky inequality and an obvious inequality

$$(a^{\frac{1}{2}}+b^{\frac{1}{2}})^2+c\leqslant\{a^{\frac{1}{2}}+(b+c)^{\frac{1}{2}}\}^2, \text{ where } a, b, c \text{ are positive.}$$

A similar inequality for $\sum_{n=0}^{\infty}\frac{\psi_{2n}^2(x)}{(\lambda_n+\mu)^{2s}}$ also holds.

If $\psi_\lambda(z)\equiv\begin{pmatrix}\psi_{1\lambda}(z)\\ \psi_{2\lambda}(x)\end{pmatrix}$ be a solution of the system $d^2u/dz^2+\lambda u=0$; $d^2v/dz^2+\lambda v=0$, satisfying either the Dirichlet or the Neumann boundary conditions, then it is easy to verify that

$$g_1^T(\xi,z,k)=-\int_0^\infty\frac{\psi_{1\lambda}(z)\,\psi_\lambda^T(\xi)}{k^2(x)+\lambda}d\rho(\lambda) \quad \dots (2.10)$$

$\rho(\lambda)$ being the spectral matrix associated with the system, where

$$k^2(x)=\mu+\Delta(x) \text{ or } \mu+\eta(x).$$

Also $(g_{ij}(\xi,z,k))$ has the explicit representation

$$\left.\begin{aligned}\left(g_{ij}(\xi,z,k)\right)&=\pm\frac{1}{2k}\left[e^{-k(z+\xi)}\mp e^{k(z-\xi)}\right]I,\quad z\leqslant\xi\\ &=\pm\frac{1}{2k}\left[e^{-k(z+\xi)}\mp e^{k(\xi-z)}\right]I,\quad z>\xi\end{aligned}\right\} \quad \dots (2.11)$$

where I is the unit matrix defined before, the upper or the lower sign being chosen according as the problem is the Dirichlet or the Neumann and $k\equiv k(x)$.

Since $g_{12}=0$, it follows from (2.10) and (2.11) and the Parseval relation that

$$\frac{1}{\{(s-1)!\}^2}\int_0^\infty\{D_\mu^{(s-1)}g_{11}(y,x,k)\}^2dy=\frac{1}{(2s-1)!}D_\mu^{(2s-1)}g_{11}(x,x,k)$$

$$=\frac{C_0}{k^{4s-1}(x)}\pm\frac{1}{(2s-1)!}D_\mu^{(2s-1)}\left(\frac{e^{-2kx}}{k}\right),$$

where $k^2\equiv k^2(x)=\mu+\Delta(x)$ or $\mu+\eta(x)$,

and $C_0=\frac{1.3.5\ldots\ldots(4s-3)}{2^{2s}(2s-1)!}$ (the upper sign being chosen for the Dirichlet problem, and the lower for the Neumann).

Since by Leibnitz's theorem,

$$D_{\mu}^{(2s-1)}\left(e^{-2kx}/k\right)=-e^{-2kx}f(k,x)\,k^{-R},$$

where $f(k, x)>0$, is a polynomial in k, x and R is a suitably chosen positive integer, it follows from (2.9) that

$$\sum_{n=0}^{\infty}\frac{\psi_{1n}^{2}(x)}{(\lambda_n+\mu)^{2s}}\leqq\left(T^{1/2}+\|J_s(x,y,k)\|_{0,\infty}^{1/2}\right)^{2} \quad \ldots(2.12)$$

where $T=C_0\,/\,k^{4s-1}(x)\pm\frac{1}{(2s-1)\,!}e^{-2kx}f(k,x)\;k^{-R}$

(the upper sign being for the Neumann, and the lower sign being for the Dirichlet problem).

$$\text{Similarly}\ \sum_{n=0}^{\infty}\frac{\psi_{2n}^{2}(x)}{(\lambda_n+\mu)^{2s}}\leqq\left(T^{1/2}+\|L_s(x,y,k)\|_{0,\infty}^{1/2}\right)^{2} \quad \ldots(2.13)$$

where $L_s(x, y, k)\equiv\begin{pmatrix}L_{1s}(x,y,k)\\ L_{2s}(x,y,k)\end{pmatrix}$ is defined in the same way as $J_s(x, y, k)$ with $g_1(\xi, x, k)$ replaced by $g_2(\xi, x, k)$.

3. Inequality involving $S_{\tau_1,\tau_2,s,m}$ and $I_{\tau_1,\tau_2,s,m}$.

Multiply (2.12) by $t_1^{\tau_1}(x)\ \Delta^{-2m}(x)$ and (2.13) by $t_2^{\tau_2}(x)\ \Delta^{-2m}(x)$, make use of the inequalities $\lambda_n+\mu\geqq\lambda_0+\mu\geqq 1$ (since $\mu\geqq 1$);

$$\left((a^{1/2}+b^{1/2})^2+(c^{1/2}+d^{1/2})\right)^2\leqq\left[a^{1/2}+c^{1/2}+(b+d)^{1/2}\right]^2$$

where $a, b, c, d\geqq 0$, and the Minkowsky inequality. Then after some reductions, we obtain

$$S^{1/2}_{\tau_1,\tau_2,s,m}$$

$$\leqslant\left[C_0\int_0^{\infty}\frac{t_1^{\tau_1}(x)+t_2^{\tau_2}(x)}{\Delta^{2m}(x)\;k^{4s-1}(x)}\,dx\right]^{1/2}$$

$$+\left[\frac{1}{(2s-1)\,!}\int_0^{\infty}\frac{e^{-2kx}f(k,x)}{k^{R}\Delta^{2m}(x)}\left(t_1^{\tau_1}(x)+t_2^{\tau_2}(x)\right)dx\right]^{1/2}$$

$$+\left[\int_0^\infty\left\{\frac{t_1^{\tau_1}(x)}{\Delta^{2m}(x)}\,\|\,J_s(x,y,k)\,\|_{0,\infty}^2+\frac{t_2^{\tau_2}(x)}{\Delta^{2m}(x)}\,\|\,L_s(x,y,k)\,\|_{0,\infty}^2\right\}dx\right]^{1/2} \quad \ldots (3.1)$$

$=I_1+I_2+I_3$, say.

Since $k^2(x)=\mu+\Delta(x)>\mu,\ \Delta(x)$, we have

$$I_1^2=I_{\tau_1,\tau_2,s,m}+C_0\int_0^\infty\frac{t_1^{\tau_1}(x)+t_2^{\tau_2}(x)}{\Delta^{2m}(x)\,k^{4s-1}(x)}\left[1-\left(\frac{\Delta(x)}{\mu+\Delta(x)}\right)^{2m}\right]dx$$

$=I_{\tau_1,\tau_2,s,m}+O(\mu^{-2s+1/2})$, as $\mu\to\infty$, by using the conditions (f) and (h).

Since $e^x>\dfrac{x^R}{R\,!}$, we have, as $\mu\to\infty$,

$$I_2^2=O\left(\int_0^\infty\frac{t_1^{\tau_1}(x)+t_2^{\tau_2}(x)}{k^{2R}(x)\,\Delta^{2m}(x)}\,dx\right)=O(\mu^{-R})\ \text{as before,}$$

by the conditions (f) and (h).

It follows from (2.8) by the application of the Minkowsky inequality, the Parseval relation and some easy reductions, that

$$\left(\int_0^\infty t_1^{\tau_1}(x)\Delta^{-2m}(x)\,\|\,J_s(x,y,k)\,\|_{0,\infty}^2\,dx\right)^{1/2}$$

$$\leqslant C\sum_{j=0}^{s-1}\left\{\sum_{n=0}^\infty\frac{1}{(\lambda_n+\mu)^{2j+2}}\int_0^\infty\frac{t_1^{\tau_1}(x)}{\Delta^{2m}(x)}\times\right.$$

$$\left.\left[\int_0^\infty D_\mu^{(s-j-1)}g_{11}(\xi,x,k)((p(\xi)-p(x))\psi_{1n}(\xi)+(r(\xi)-r(x))\psi_{2n}(\xi))\,d\xi\right]^2dx\right\}^{1/2}$$

$$\ldots (3.2)$$

where $C=\max C_{js}$, the constants involved in (2.8).

Put $a_n^2=\displaystyle\int_0^\infty\frac{t_1^{\tau_1}(x)}{\Delta^{2m}(x)}\times$

$$\left[\int_0^\infty D_\mu^{(s-j-1)}g_{11}(\xi,x,k)\{(p(\xi)-p(x))\psi_{1n}(\xi)+(r(\xi)-r(x))\psi_{2n}(\xi)\}\,d\xi\right]^2dx.$$

$b_n = \frac{1}{(\lambda_n+\mu)^{2j+2}}$, $\alpha = \frac{2j+2m+2}{2m}$, $\beta = \frac{2j+2m+2}{2j+2}$ in the inequality

$$\sum_{n=0}^{\infty} a_n^2 b_n \leqslant \left(\sum_{n=0}^{\infty} a_n^2\right)^{1/\alpha} \left(\sum_{n=0}^{\infty} a_n^2 b_n^{\beta}\right)^{1/\beta}, \quad \frac{1}{\alpha}+\frac{1}{\beta}=1, \quad \alpha, \beta>0 \quad \ldots \text{(A)}$$

a variant of the Hölder inequality, so as to obtain from (3.2)

$$\sum_{n=0}^{\infty} \frac{a_n^2}{(\lambda_n+\mu)^{2j+2}} \leqslant \left(\sum_{n=0}^{\infty} \frac{a_n^2}{(\lambda_n+\mu)^{2j+2m+2}}\right)^{\frac{2j+2}{2j+2m+2}} \left(\sum_{n=0}^{\infty} a_n^2\right)^{\frac{2m}{2j+2m+2}} \quad \ldots (3.3)$$

Now $\widetilde{M} = \int_0^{\infty} \left[D_{\mu}^{(s-j-1)} g_{11}(\xi, x, k)\right]^2 \left[p(\xi)-p(x)\right]^2 d\xi$

$$= \left(\int_0^{x-1} + \int_{x+1}^{\infty}\right) \{p^2(\xi)-2p(x)p(\xi)+p^2(x)\} \left(D_{\mu}^{(s-j-1)} g_{11}(\xi, x, k)\right)^2 d\xi$$

$$+ \int_{x-1}^{x+1} \{p(\xi)-p(x)\}^2 \left(D_{\mu}^{(s-j-1)} g_{11}(\xi, x, k)\right)^2 d\xi$$

$= M_1+M_2+M_3+M_4$, say $(x \geqslant 1)$.

We evaluate $D_{\mu}^{(s-j-1)} g_{11}(\xi, x, k)$ by the Leibnitz formula from the explicit relations for $g_{11}(\xi, x, k)$ as given by (2.11), utilize the conditions (a), as and when necessary. Then for both the Dirichlet and the Neumann problem and for the cases $\xi>x$, $\xi<x$, it is easy to derive after some steps that

$$M_4 \leqslant d(k(x))^{-4(s-j)-1+4a_0}$$

for $x \geqslant 1$, d being a suitable positive constant.

An utilization of the conditions (b) along with the use of the explicit expression for $D_{\mu}^{(s-j-1)} g_{11}(\xi, x, k)$ as derived before by the use of the Leibnitz formula, yields after some elaborate steps, M_1, M_2, $M_3 < Ck^{-\sigma}$, where C is a positive constant and $\sigma > 0$ may be chosen as large as possible.

It therefore follows that $\widetilde{M} \leqslant d(k(x))^{-4(s-j)-1+4a_0}$.

A similar estimate holds for the expression $\widetilde{N}$, obtained from $\widetilde{M}$ by replacing $p(u)$ by $r(u)$, $u=x, \xi$.

Hence by the Bessel inequality and the estimates derived above,

$$\sum_{n=0}^{\infty}\left[\int_0^{\infty} D_{\mu}^{(s-j-1)} g_{11}(\xi, x, k)\ \{(p(\xi)-p(x))\psi_{1n}(\xi)+(r(\xi)-r(x))\psi_{2n}(\xi)\}\ d\xi\right]^2$$

$$\leqslant 2d(k(x))^{-4(s-j)-1+4a_0} \qquad \ldots (3.4)$$

It follows, therefore, that

$$\sum_{n=0}^{\infty} a_n^2 \leqslant 2d\int_0^{\infty} t_1^{\tau_1}(x)\ \Delta^{-2m}(x)\ (k(x))^{-4(s-j)-1+4a_0}\ dx \qquad \ldots (3.5)$$

$$=O(\mu^{-(5/2-2a_0)})$$

as $\mu \to \infty$, by using the conditions (f) and (h).

Let $S_N^{(j)}(x)$ denote the Nth partial sum of the infinite series

$$\sum_{n=0}^{\infty} \frac{\Delta^{-2m}(x)}{(\lambda_n+\mu)^{2j+2m+2}} \times$$

$$\left[\int_0^{\infty} D_{\mu}^{(s-j-1)} g_{11}(\xi, x, k)\{(p(\xi)-p(x))\psi_{1n}(\xi)+(r(\xi)-r(x))\psi_{2n}(\xi)\}d\xi\right]^2$$

(N=1, 2, 3, ...).

Putting

$$a_n=\int_0^{\infty} D_{\mu}^{(s-j-1)} g_{11}(\xi, x, k)\{(p(\xi)-p(x))\ \psi_{1n}(\xi)+(r(\xi)-r(x))\ \psi_{2n}(\xi)\}d\xi$$

$$b_n=(\lambda_n+\mu)^{-2j-2m-2},\ \alpha=\frac{s+m}{s-j-1},\ \beta=\frac{s+m}{j+m+1}\ \ (i<s-1)$$

in the inequality (A), we have $S_N^{(j)}(x)$

$$\leqslant \Delta^{-2m}(x)\left(\sum_{n=0}^{N}\left[\int_0^{\infty} D_{\mu}^{(s-j-1)} g_{11}(\xi, x, k)\{(p(\xi)-p(x))\ \psi_{1n}(\xi) + (r(\xi)-r(x))\ \psi_{2n}(\xi)\}d\xi\right]^2\right)^{\frac{s-j-1}{s+m}}$$

$$\cdot\left(\sum_{n=0}^{N}(\lambda_n+\mu)^{-2s-2m}\left[\int_0^{\infty} D_{\mu}^{(s-j-1)} g_{11}(\xi, x, k)\{(p(\xi)-p(x))\ \psi_{1n}(\xi) + (r(\xi)-r(x))\psi_{2n}(\xi)\}d\xi\right]^2\right)^{\frac{j+m+1}{s+m}} \qquad \ldots (3.6)$$

Now

$$Q=\int_0^\infty D_\mu^{(s-j-1)} g_{11}(\xi, x, k)\{p(\xi)-p(x)\}\, \psi_{1n}(\xi)d\xi$$

$$=\int_{x-1}^{x+1} D_\mu^{(s-j-1)} g_{11}(\xi, x, k)\{p(\xi)-p(x)\}\, \psi_{1n}(\xi)d\xi$$

$$+\left(\int_0^{x-1}+\int_{x+1}^\infty\right)p(\xi)\, D_\mu^{(s-j-1)} g_{11}(\xi, x, k)\, \psi_{1n}(\xi)d\xi$$

$$-p(x)\left(\int_0^{x-1}+\int_{x+1}^\infty\right)D_\mu^{(s-j-1)} g_{11}(\xi, x, k)\psi_{1n}(\xi)d\xi$$

$=Q_1+Q_2+Q_3$, say $(x \geqslant 1)$.

Evaluating $D_\mu^{(s-j-1)} g_{11}(\xi, x, k)$ by the Leibnitz formula, from the explicit relations for $g_{11}(\xi, x, k)$ from (2.11), we obtain both for the Dirichlet and the Neumann problems, on using the conditions (a), for all $x \geqslant 1$,

$$Q_1^2<\frac{C}{\{k(x)\}^{4(s-j)-4a_0+2-2\epsilon}}\left(\int_0^1 \gamma^{\epsilon-1}\, |\, \psi_{1n}(x-\gamma)\, |\, d\gamma\right)^2 \qquad \dots (3.7)$$

where ϵ is an arbitrarily chosen positive number, which can be taken sufficiently small.

It is easy to verify by using the conditions (b), that Q_2^2, $Q_3^2<\frac{C}{\{k(x)\}^\sigma}$ for any positive number σ, however large. Therefore,

$$Q^2 \leqslant 3\left(Q_1^2+Q_2^2+Q_3^2\right)$$

$$<C\left\{\frac{1}{\{k(x)\}^{4(s-j)-4a_0+2-2\epsilon}}\left[\left(\int_0^1 \gamma^{\epsilon-1}\, |\, \psi_{1n}(x-\gamma)\, |\, d\gamma\right)^2+\{k(x)\}^{-\sigma}\right]\right\}$$

A similar estimate holds for the expression Q^*, obtained from Q by replacing $p(v)$ by $r(v)$, $v=x, \xi$. Hence from the relations (3.4) and (3.6), for any $j=0, 1, 2, \dots s-2$, $x \geqslant 1$,

$$S_N^{(j)}(x) < \frac{C}{\Delta^{2m}(x)} \left\{ \sum_{n=0}^{N} \frac{k(x)^{-4(s-j)+4a_0-2+2\epsilon}}{(\lambda_n+\mu)^{2s+m}} \times \right.$$

$$\left[\left(\int_0^1 \gamma^{\epsilon-1} \mid \psi_{1n}(x-\gamma) \mid d\gamma\right)^2 + k(x)^{-\sigma}\right]\Bigg\}^{\frac{j+m+1}{s+m}} . \left\{ k(x)^{-4(s-j)+4a_0-1} \right\}^{\frac{s-j-1}{s+m}}$$

$$= \frac{C}{\Delta^{2m}(x)} \left\{ k(x)^{-\nu} \right\}^{\frac{s-j-1}{s+m}} \left\{ \sum_{n=0}^{N} \frac{1}{(\lambda_n+\mu)^{2s+2m}} \times \right.$$

$$\left[\left(\int_0^1 \gamma^{\epsilon-1} \mid \psi_{1n}(x-\gamma) \mid d\gamma\right)^2 + k(x)^{-\sigma}\right]\Bigg\}^{\frac{j+m+1}{s+m}} \qquad \dots (3.8)$$

where

$$\nu = 4(s-j) - 4a_0 + 1 + \frac{4(s-j) - 4a_0 + 2 - 2\epsilon}{s-j-1}(j+m+1) > 4(s+m) - 1 + 2\omega(s+m)$$

ω being a positive number determined by $6 - 4a_0 - 2\epsilon > 2\omega(s-1)$ ... (3.9)

Multiply the relations (3.8) by $t_1^{\tau_1}(x)$ and integrate between the limits $(1, \infty)$ and $(0,1)$ separately. Then by adopting the analysis of Levitan and and Sargsyan [4] in dealing with the part involving the term $\psi^2_{1n}(x-\gamma)$ we obtain after some elaborate computations, the inequality

$$\int_0^\infty t_1^{\tau_1}(x)\, S_N^{(j)}(x)\, dx$$

$$< C\mu^{-\omega} \left(\int_0^\infty \frac{t_1^{\tau_1}(x) \Delta^{-2m}(x)}{k^{4(s+m)-1}(x)} dx \right)^{\frac{i-m-1}{s+m}} . \left(\sum_{n=0}^{N} \frac{a_{1n}^{(\tau_1)}}{(\lambda_n+\mu)^{2s+2m}} \right)^{\frac{j+m+1}{s+m}} \qquad \dots (3.10)$$

where $a_{1n}^{(\tau_1)} = \int_0^\infty \frac{t_1^{\tau_1}(x)\, \psi^2_{1n}(x)}{\Delta^{2m}(x)} dx$, and $j = 0, 1, 2, \dots, s-2$.

Since $\quad a^{1/\alpha}\, b^{1/\beta} \leqslant a+b, \quad$ where $a, b > 0$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ ... (B)

it follows from (3.10) that

$$\int_0^\infty t_1^{\tau_1}(x)\, S_N^{(j)}(x)\, dx < C\mu^{-\omega} \left(\int_0^\infty \frac{t_1^{\tau_1}(x)\, \Delta^{-2m}(x)}{k^{4(s+m)-1}(x)}\, dx + \sum_{n=0}^{N} \frac{a_{1n}^{(\tau_1)}}{(\lambda_n+\mu)^{2s+2m}} \right)$$

... (3.11)

for $j = 0, 1, 2, \dots, s-2$.

Also, it can be verified similarly that the relation (3.11) is valid for $j=s-1$.

Now $$\sum_{n=0}^{N}\frac{a_n^2}{(\lambda_n+\mu)^{2j+2m+2}}=\int_0^\infty t_1^{\tau_1}(x)\, S_N^{(j)}(x)\, dx,$$

for any positive integer N.

Therefore from (3.2), (3.3), (3.5) and (3.11), we obtain

$$\left(\int_0^\infty \frac{t_1^{\tau_1}(x)}{\Delta^{2m}(x)}\, \| J_s(x, y, k) \|_{0\,\infty}\, dx\right)^{1/2}$$

$$=\mu^{-\frac{m(5-4a_0)}{4(s+m)}}\, O\left(\int_0^\infty \frac{t_1^{\tau_1}(x)\, dx}{k^{4(s+m)-1}(x)}+\sum_{n=0}^{\infty}\frac{a_{1n}^{(\tau_1)}}{(\lambda_n+\mu)^{2s+2m}}\right)^{\frac{1}{2}} \qquad \text{... (3.12)}$$

Similarly

$$\left(\int_0^\infty \frac{t_2^{\tau_2}(x)}{\Delta^{2m}(x)}\, \| L_s(x, y, k) \|_{0,\infty}\, dx\right)^{1/2}$$

$$=\mu^{-\frac{m(5-4a_0)}{4(s+m)}}\, O\left(\int_0^\infty \frac{t_2^{\tau_2}(x)\, dx}{k^{4(s+m)-1}(x)}+\sum_{n=0}^{\infty}\frac{a_{2n}^{(\tau_2)}}{(\lambda_n+\mu)^{2s+2m}}\right)^{1/2} \qquad \text{... (3.13)}$$

where

$$a_{2n}^{(\tau_2)}=\int_0^\infty \frac{t_2^{\tau_2}(x)\, \psi_{2n}^2(x)}{\Delta^{2m}(x)}\, dx.$$

From (3.1), (3.12) and (3.13) using the estimates of I_1^2, I_2^2, it follows that

$$S^{\frac{1}{2}}_{\tau_1,\tau_2,s,m} \leq I^{\frac{1}{2}}_{\tau_1,\tau_2,s,m} +O\left(\mu^{-s+\frac{1}{4}}\right)+O\left(\mu^{-R/2}\right)$$

$$+\mu^{-\frac{m(5-4a_0)}{4(s+m)}}\, O\left(I^{\frac{1}{2}}_{\tau_1,\tau_2,s,m}+S^{\frac{1}{2}}_{\tau_1,\tau_2,s,m}\right) \qquad \text{... (3.14)}$$

as $\mu\to\infty$.

From this, it follows that

$$\lim_{\mu\to\infty}\frac{S_{\tau_1,\tau_2,s,m}}{I_{\tau_1,\tau_2,s,m}}\leq 1 \qquad \text{... (3.15)}$$

When $m=0$, it follows as in Levitan, Sargsyan [4], that the inequality corresponding to (3.14) is

$$S^{\frac{1}{2}}_{\tau_1,\tau_2,s} \leq I^{\frac{1}{2}}_{\tau_1,\tau_2,s}+C\mu^{-\omega/2}\left(I^{\frac{1}{2}}_{\tau_1,\tau_2,s}+S^{\frac{1}{2}}_{\tau_1,\tau_2,s}\right) \qquad \text{... (3.16)}$$

where $S_{\tau_1,\tau_2,s}=S_{\tau_1,\tau_2,s,0}$; $I_{\tau_1,\tau_2,s}=I_{\tau_1,\tau_2,s,0}$, ω is determined by (3.9) and the condition (h) is replaced by the condition (h′). The result is that $S_{\tau_1,\tau_2,s}$ is convergent, if $I_{\tau_1,\tau_2,s}$ is convergent.

4. A reverse inequality connecting $S_{\tau_1,\tau_2,s,m}$ and $I_{\tau_1,\tau_2,s,m}$.

From (2.7)

$$\frac{1}{(s-1)\,!}D_\mu^{(s-1)} g_1(y, x, k) = \frac{1}{(s-1)\,!} D_\mu^{(s-1)} G_1^T{}_1(x, y, \mu) - J_s(x, y, k).$$

We first apply Minkowski's inequality and substitute the value of $\int_0^\infty \left(D_\mu^{(s-1)} g_{11}\right)^2 dy$, as derived in art. 2. Then multiplying both sides of the result so obtained by $t_1^{\tau_1}(x)$, we integrate between the limits $[0, \infty)$, and apply the Minkowsky inequality again. This yields

$$\left[C_0\int_0^\infty \frac{t_1^{\tau_1}(x)\,dx}{(\mu+\Delta)^{2(s+m)-\frac{1}{2}}} \pm \int_0^\infty \frac{e^{-2kx} f(k, x) t_1^{\tau_1}(x)}{k^R(2s-1)\,!\,(\mu+\Delta)^{2m}}\,dx\right]^{1/2}$$

$$\leq \left(\int_0^\infty \sum_{n=0}^\infty \frac{t_1^{\tau_1}(x)\,\psi_{1n}^2(x)}{\Delta^{2m}(x)(\lambda_n+\mu)^{2s}}\,dx\right)^{1/2} + \left(\int_0^\infty \frac{t_1^{\tau_1}(x)}{\Delta^{2m}(x)} \| J_s(x, y, k) \|_{0,\infty}\, dx\right)^{1/2} \qquad \dots (4.1)$$

Similarly

$$\left[C_0\int_0^\infty \frac{t_2^{\tau_2}(x)\,dx}{(\mu+\Delta)^{2(s+m)-\frac{1}{2}}} \pm \int_0^\infty \frac{e^{-2kx} f(k, x)\, t_2^{\tau_2}(x)}{k^R(2s-1)\,!\,(\mu+\Delta)^{2m}}\,dx\right]^{1/2}$$

$$\leq \left(\int_0^\infty \sum_{n=0}^\infty \frac{t_2^{\tau_2}(x)\,\psi_{2n}^2(x)}{\Delta^{2m}(x)(\lambda_n+\mu)^{2s}}\,dx\right)^{1/2} + \left(\int_0^\infty \frac{t_2^{\tau_2}(x)}{\Delta^{2m}(x)} \| L_s(x, y, k) \|_{0,\infty}\, dx\right)^{1/2} \qquad \dots (4.2)$$

From (4.1) and (4.2), using the relations (3.12) and (3.13), and the inequality

$$(a+c)^{1/2} \leq (b+d)^{1/2} + (u+v)^{1/2}$$

when $\quad a^{1/2} \leq b^{1/2} + u^{1/2},\ c^{1/2} \leq d^{1/2} + v^{1/2},\ a, b, c, d, u, v \geq 0,$

we obtain

$$\{I_{\tau_1,\tau_2,s,m} + O(\mu^{-R})\}^{1/2}$$

$$\leq \left(\sum_{n=0}^\infty \frac{a_n^{(\tau_1,\tau_2,m)}}{(\lambda_n+\mu)^{2s}}\right)^{1/2} + \mu^{-\left[\frac{m(5-4a_0)}{4(s+m)}\right]} O\left(I_{\tau_1,\tau_2,s,m} + S_{\tau_1,\tau_2,s,m}\right)^{1/2} \qquad \dots (4.3)$$

To find the estimate of $\sum_{n=0}^{\infty} \frac{a_n^{(\tau_1, \tau_2, m)}}{(\lambda_n+\mu)^{2s}}$

put $a_n^2 = \frac{a_n^{(\tau_1, \tau_2, m)}}{(\lambda_n+\mu)^{2s-2c}}$, $b_n = \frac{1}{(\lambda_n+\mu)^{2c}}$, $\alpha = \frac{2m+2c}{2m}$, $\beta = \frac{2m+2c}{2c}$

$0 < c \leqq 2$, in the inequality (A). Then after reduction, by using the inequality (B), if follows that

$$\sum_{n=0}^{\infty} \frac{a_n^{(\tau_1, \tau_2, m)}}{(\lambda_n+\mu)^{2s}} \leqq \mu^{-c} \sum_{n=0}^{\infty} \frac{a_n^{(\tau_1, \tau_2, m)}}{(\lambda_n+\mu)^{2s-c}} + S_{\tau_1, \tau_2, s, m} \qquad \ldots (4.4)$$

Now $\sum_{n=0}^{\infty} \frac{a_n^{(\tau_1, \tau_2, m)}}{(\lambda_n+\mu)^{2s-c}} \leqq \sum_{n=0}^{\infty} \frac{a_n^{(\tau_1, \tau_2)}}{(\lambda_n+\mu)^{2(s-1)}}$

which is convergent if $\int_0^{\infty} \frac{t_1^{\tau_2}(x) + t_2^{\tau_2}(x)}{\{\mu+\Delta(x)\}^{2(s-1)-\frac{1}{2}}} dx$ is convergent.

Here $a_n^{(\tau_1, \tau_2)} = a^{(\tau_1, \tau_2, 0)}$.

But the integral is convergent by virtue of the condition (h).

From (4.3) and (4.4), therefore it follows as before that

$$\lim_{\mu\to\infty} \frac{I_{\tau_1, \tau_2, s, m}}{S_{\tau_1, \tau_2, s, m}} \leqq 1 \qquad \ldots (4.5)$$

From (3.15) and (4.5) we ultimately obtain

$$S_{\tau_1 \tau_1, s, m} \sim I_{\tau_1, \tau_2, s, m}, \quad \text{as } \mu\to\infty \qquad \ldots (4.6)$$

When $k^2(x) = \mu + \eta(x)$, the procedure is exactly similar, but we have now to utilize the condition (c), in addition. It then follows that

$$S^*_{\tau_1, \tau_2, s, m} \sim I^*_{\tau_1, \tau_2, s, m} \quad \text{as } \mu\to\infty \qquad \ldots (4.7)$$

We can interchange $S_{\tau_1, \tau_2, s}$ and $I_{\tau_1, \tau_2, s}$ in (3.16), in the Minkowsky inequality involved in the process of its derivations, so as to obtain ultimately

$$S_{\tau_1, \tau_2, s} \sim I_{\tau_1, \tau_2, s} \text{ as } \mu\to\infty .$$

Similarly $S^*_{\tau_1, \tau_2, s} \sim I^*_{\tau_1, \tau_2, s}$ as $\mu\to\infty$; we have to use the condition (c) in addition, in the process of derivation of this result.

Altogether (4.6) and (4.7) hold for $m=0$ when the condition (h) is replaced by the condition (h').

Finally, it remains an open question whether the asymptotic relations (4.6), (4.7) still hold when

$$0 < m < \max\left(\frac{A_1}{2}, \frac{A_2}{2}\right).$$

5. Application of the Tauberian theorem

In the following we utilize the Tauberian theorem due to Korenlum [3] viz,

If $f(x) = \int_0^\infty K^*(\xi/x)d\,\phi(\xi)$, $g(x) = \int_0^\infty K^*(\xi/x)d\,\psi(\xi)$ and if $g(x)/f(x) \to 1$ as $x \to \infty$, then $\psi(x)/\phi(x)$ also tends to 1, as x tends to infinity, provided

(i) $\psi(x)$, $\phi(x)$ are non-negative, non-decreasing functions of $x > 0$, where $\lim\limits_{x\to\infty} \phi(x) = \infty$,

(ii) $\alpha\,\phi(x) < x\phi'(x) < \beta\,\phi(x)$ where $0 < \beta < \alpha + 1$ and x is large enough,

or (iii) $\dfrac{\phi(y)}{\phi(x)} \leqslant \left(\dfrac{y}{x}\right)^{\gamma}$ for large x and $y > x$, where γ is some positive number.

In the special case we can choose γ to be unity and this condition is satisfied if ϕ is convex downwards.

(iv) $K^*(x) \geqslant 0$, $0 < x < \infty$, $K^*(+0) > 0$, $K^*(x) = O(x^{-\nu})$ as $x \to \infty$,

$\int_0^\infty | K^{*\prime}(x) | (1+x^{\nu})\,dx < \infty$ and $\int_0^\infty K^{*\prime}(xt)\, S(t) = 0$, for the class of functions $S(x)$, bounded on each finite interval with $S(x) = O(x^{\nu})$ as $x \to \infty$, implies $S(x) \equiv 0$ identically.

In our case the kernel is $K^*(x) = (1+x)^{-as-am}$ which satisfies all the conditions stated in (iv).

Put $t_1(x) = t_2(x) = \Delta(x) \equiv \Delta$ (or η).

Let $I'_{\tau_1, \tau_2, s, m}$ be the $I_{\tau_1, \tau_2, s, m}$ of Art. 1, in which t_j are replaced by Δ ; with a similar meaning for $S'_{\tau_1, \tau_2, s, m}$.

Then by using the Legendre duplication formula,

$$I'_{\tau_1, \tau_2, s, m}=\frac{\Gamma(2s-\frac{1}{2})}{2\sqrt{\pi}\,\Gamma(2s)}\int_0^\infty \frac{\Delta^{\tau_1}(x)+\Delta^{\tau_2}(x)}{(\mu+\Delta)^{2(s+m)-1/2}}\,dx$$

and

$$S'_{\tau_1, \tau_2, s, m}=\sum_{n=0}^\infty \frac{A_n^{(\tau_1, \tau_2, m)}}{(\lambda_n+\mu)^{2s+2m}}, \text{ where } A_n^{(\tau_1, \tau_2, m)}=\int_0^\infty \sum_{j=1}^2 \Delta^{\tau_j-2m}\,\psi_{jm}^2\,dx.$$

Let $\sigma(\lambda)=\text{mes}\{\Delta(x)<\lambda\}$. Then

$$\int_0^\infty \frac{\Delta^{\tau_j}(x)\,dx}{(\mu+\Delta)^{2s+2m-1/2}}=\int_0^\infty \frac{\lambda^{\tau_j}\,d\sigma(\lambda)}{(\mu+\Delta)^{2s+2m-1/2}},\ j=1, 2 \qquad \ldots (5.1)$$

Put $\psi_{\tau_j}(\lambda)=\int_0^\lambda (\lambda-u)^{1/2}\,u^{\tau_j}\,d\sigma(u)$, so that

$$\int_0^\infty \frac{d\psi_{\tau_j}(\lambda)}{(\lambda+\mu)^{2s+2m}}=\frac{1}{2}\int_0^\infty \frac{d\lambda}{(\lambda+\mu)^{2s+2m}}\int_0^\lambda (\lambda-u)^{-1/2}\,u^{\tau_j}\,d\sigma(u)$$

$$=\frac{1}{2}\sqrt{\pi}\,\frac{\Gamma(2s+2m-\frac{1}{2})}{\Gamma(2s+2m)}\int_0^\infty \frac{\Delta^{\tau_j}(x)\,dx}{\{\mu+\Delta(x)\}^{2s+2m-1/2}},$$ on changing the order of integration, subsequent reduction and utilization of the relations (5.1). Thus

$$\int_0^\infty \frac{d\psi_{\tau_1, \tau_2}(\lambda)}{(\lambda+\mu)^{2s+2m}}=\frac{\pi\Gamma(2s)\,\Gamma(2s+2m-\frac{1}{2})}{\Gamma(2s-\frac{1}{2})\,\Gamma(2s+2m)}\,I'_{\tau_1, \tau_2, s, m}$$

where $\psi_{\tau_1, \tau_2}(\lambda)=\psi_{\tau_1}(\lambda)+\psi_{\tau_2}(\lambda)$.

Again let $\chi_{\tau_1, \tau_2}(\lambda)=\sum_{\lambda_n<\lambda} A_n^{(\tau_1, \tau_2, m)}$, so that

$$\int_0^\infty d\chi_{\tau_1, \tau_2}(\lambda)\,(\lambda+\mu)^{-2s-2m}=S'_{\tau_1, \tau_2, s, m}.$$

Therefore from (4.6)

$$\int_0^\infty \frac{d\chi_{\tau_1,\tau_2}(\lambda)}{(\lambda+\mu)^{2s+2m}} \sim \frac{\Gamma(2s+2m)\,\Gamma(2s-\frac{1}{2})}{\pi\,\Gamma(2s)\,\Gamma(2s+2m-\frac{1}{2})}\int_0^\infty \frac{d\psi_{\tau_1,\tau_2}(\lambda)}{(\lambda+\mu)^{2s+2m}} \quad \ldots (5.2)$$

as $\mu\to\infty$.

Hence by the Korenlum Tauberian theorem, it follows that

$$\chi_{\tau_1,\tau_2}(\lambda) \sim \frac{1}{\pi}\frac{\Gamma(2s+2m)\Gamma(2s-\frac{1}{2})}{\Gamma(2s)\Gamma(2s+2m-\frac{1}{2})}\psi_{\tau_1,\tau_2}(\lambda), \text{ as } \lambda\to\infty.$$

That is

$$\sum_{\lambda_n<\lambda} A_n^{(\tau_1,\tau_2,m)} \sim \frac{1}{\pi}\frac{\Gamma(2s+2m)\,\Gamma(2s-\frac{1}{2})}{\Gamma(2s)\,\Gamma(2s+2m-\frac{1}{2})}\int_{\Delta(x)<\lambda}(\Delta^{\tau_1}+\Delta^{\tau_2})(\lambda-\Delta)^{1/2}dx \quad (5.3)$$

as λ tends to infinity.

The condition (ii) of the Korenlum Tauberian theorem in the present case is

$$\alpha\psi_{\tau_1,\tau_2}(x) < x\psi'_{\tau_1,\tau_2}(x) < \beta\,\psi_{\tau_1,\tau_2}(x) \text{ for large } x,\ 0<\beta<\alpha+1.$$

Alternatively, the condition (iii) (with $\gamma=1$) is satisfied in the present case, if $\psi_{\tau_1,\tau_2}(\lambda)$ is convex downwards for which it is necessary and sufficient that $\psi''_{\tau_1,\tau_2}(\lambda)\geqq 0$. A sufficient condition for the validity of this relation is Δ is three times differentiable, with

$$\tau_j(\tau_j-1)\Delta'^4-2\tau_j\Delta\Delta'^2\Delta''+\Delta^2(3\Delta''^2-\Delta'\Delta''')\geqq 0,\ (j=1,2),\ \Delta\equiv\Delta(x), \quad \ldots (5.3a)$$

far all large x, as can be verified by actually differentiating twice the integral expression for $\psi_{\tau_1,\tau_2}(\lambda)$ involving $t(u)$, the inverse function of $\Delta(x)=u$ as in Levitan and Sargsyan [4] and then twice integrating by parts.

Similarly

$$\sum_{\lambda_n<\lambda} B_n^{(\tau_1,\tau_2,m)} \sim \frac{1}{\pi}\frac{\Gamma(2s+2m)\Gamma(2s-\frac{1}{2})}{\Gamma(2s)\Gamma(2s+2m-\frac{1}{2})}\times$$

$$\int_{\eta(x)<\lambda}\left(\eta^{\tau_1}+\eta^{\tau_2}\right)\left(\lambda-\eta\right)^{1/2}dx \quad \ldots (5.4)$$

as $\lambda\to\infty$, where $B_n^{(\tau_1,\tau_2,m)}=\int_0^\infty\left(\eta^{\tau_1-2m}\psi_{1n}^2+\eta^{\tau_2-2m}\psi_{2n}^2\right)dx$, provided a relation similar to (5.3a), in which Δ is replaced by η, is satisfied.

The asymptotic relations (5.3), (5.4) hold, when $\tau_1=\tau_2=m=0$, so that

$$N(\lambda)\sim\frac{2}{\pi}\int_{\Delta(x)<\lambda}(\lambda-\Delta)^{1/2}\,dx \text{ and also } N(\lambda)\sim\frac{2}{\pi}\int_{\eta(x)<\lambda}(\lambda-\eta)^{1/2}\,dx, \text{ as } \lambda\to\infty.$$

Thus

$$N(\lambda)\sim\frac{1}{\pi}\left[\int_{\Delta(x)<\lambda}\{\lambda-\Delta(x)\}^{1/2}\,dx+\int_{\eta(x)<\lambda}\{\lambda-\eta(x)\}^{1/2}\,dx\right], \text{ as } \lambda\to\infty.$$

REFERENCES

[1] Chakravarty, N. K.—Some problems in eigenfunction expansions (I), *Quart. J. Math. (Oxford)*, 16, 1965, 135-150.

[2] Chakravarty N. K. and Sengupta P. K.—On the distribution of eigenvalues of a matrix differential operator, *Journ. Ind. Inst. Sci.* 61 (B), 1979, 19-42.

[3] Korenlum B. I.—General Tauberian theorem for quotient functions, *Dokladi Acad. Nauk. SSSR*, (5), LXXXVIII, 1953, 745-748.

[4] Levitan B. M. and Sargsyan I. S.—Some problems in the theory of Sturm-Liouville equations, Russian Mathematical Surveys (15) (1), 1960, 1-95.

[5] Sengupta P. K.—On the convergence of eigenfunction expansions etc.—*Journ. Ind. Inst. Sci.*, 60(3), 1978, 89-110.

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