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A NEW CLASS OF GENERATING RELATIONS INVOLVING LAGUERRE AND GEGENBAUER POLYNOMIALS

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1. Introduction : Recently unified theory for obtaining generating functions of special functions has received much attention owing to the fact that a number of particular bilateral (or bilinear) generating functions for some special functions, for example the well known Hille-Hardy formula for Laguerre polynomials, Mehler's formula for Hermite polynomials and others, are available in the literature. Such unified theory ,consists in deriving a class of bilateral or trilateral generating functions for special functions whereby the particular bilateral or trilateral generating functions will follow easily as consequences of the theory. A question in this direction was raised by C. Truesdell [7] and some answers are given by W. A. Al-Salam [1] and S. K. Chatterjea [3,4,] in connection with bilateral generating functions involving Laguerre, Hermite and Gegenbauer polynomials in recent years. It is interesting to note that the particular bilateral generating relation of L. Weisner [8], viz.

$$(1.1)\quad \rho^{-2\lambda} \exp\left[\frac{-yt(x-t)}{\rho^2}\right] {}_0F_1\left[-\,;\lambda+\tfrac{1}{2}, \frac{y^2t^2(x^2-1)}{4\rho^4}\right]$$

$$=\sum_{r=0}^{\infty} r\,!\, L_r^{(2\lambda-1)}(y)\, C_r^{\lambda}(x)\frac{t^r}{(2\lambda)_r}\,;\quad \rho=(1-2xt+t^2)^{1/2}$$

follows at once from the following theorem of Chatterjea on a class of bilateral generating functions for Gegenbauer polynominals :

If there exists a unilateral generating relation of the form

$$(1.2)\quad F(x,t)=\sum_{m=0}^{\infty} a_m t^m C_m^{\lambda}(x)$$

then there will exist a bilateral generating relation of the form

$$(1.3)\quad \rho^{-2\lambda}\, F\left(\frac{x-t}{\rho}, \frac{ty}{\rho}\right)=\sum_{r=0}^{\infty} t^r b_r(y) C_r^{\lambda}(x)$$

where

$$b_r(y)=\sum_{m=0}^{r}\binom{r}{m}a_m y^m.$$

The importance of this class of bilateral generating relations lies in the fact that one can at once derive a large number of bilateral generating relations for Gegenbauer polynomials by attributing different values to a_m.

Now in the investigation of such class of generating relations, group theoretic-method seems to be a potent one in comparison with analytic method, because the unknown generating function can only be obtained by group theoretic-method, whereas the known generating function can be verified and then extended by analytic method. As an illustration of this statement we may cite the following extension of Mehler's formula by Chatterjea [5] :

$$(1.4)\ \sum_{k=0}^{\infty}\frac{t^k}{2^k k!}\sum_{r=0}^{n}2^r r!\binom{n}{r}\binom{n}{r}\left(\frac{-t}{1-t^2}\right)^r H_{k+n-r}(x)\,H_{k+n-r}(y)$$

$$=\left(1-t^2\right)^{-(n+\frac{1}{2})}\exp\left[\frac{2xyt-(x^2+y^2)t^2}{1-t^2}\right]H_n\left(\frac{x-yt}{\sqrt{1-t^2}}\right)H_n\left(\frac{y-xt}{\sqrt{1-t^2}}\right),$$

which could not be derived by analytic method prior to its existence. In fact, the very nature of Chatterjea's formula helps L. Carlitz [2] to extend further.

So in the present paper we shall adopt group-theoretic method to obtain a new class of mixed trilateral generating relation involving Laguerre and Gegenbauer polynomials. We shall use the raising operators of the Lie algebras in connection with Laguerre and Gengenbauer polynomials [6]. Our main theorem can be stated in the following form :

Theorem : If there exists a bilateral generating relation of the form

$$(1.5)\quad G(x,z,w)=\sum_{n=0}^{\infty}a_n L_n^{(\alpha)}(x)C_n^{\lambda}(z)w^n$$

then there exists a mixed trilateral generating relation

$$(1-w)^{-\alpha-1}\,(1-2wz+w^2)^{-\lambda}\exp\left(-\frac{wx}{1-w}\right).$$

$$\cdot\, G\left(\frac{x}{1-w},\ \frac{z-w}{\sqrt{w^2-2wz+1}},\ \frac{wv}{(1-w)\sqrt{w^2-2wz+1}}\right)$$

$$(1.6)\quad =\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f_{m,n}(w,v,x)\,C_n^{\lambda}(z)$$

where

$$f_{m,n}(w,v,x)=\sum_{p=0}^{\min(m,n)}\binom{m+n-2p}{n-p}\binom{n}{p}a_{n-p}\,w^{m+n-p}\,v^{n-p}\,L_{n+m-2p}^{(\alpha)}(x)$$

and $\binom{n}{p}=\dfrac{n!}{p!\,(n-p)!}$.

The above theorem is illustrated by means of a well known bilateral generating relation (1.1) due to L. Weisner.

2. Group-theoretic method: For the Laguerre polynomials $L_n^{(\alpha)}(x)$ defined by

$$(2.1)\quad (1-t)^{-\alpha-1}\exp\left(\frac{-xt}{1-t}\right)=\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\, t^n,$$

we consider the operator R_1, where

$$(2.2)\quad R_1=xy\frac{\partial}{\partial x}+y^2\frac{\partial}{\partial y}+(-x+\alpha+1)y$$

such that

$$(2.3)\quad R_1[F_n^{(\alpha)}(x,y)]=(n+1)\,F_{n+1}^{(\alpha)}(x,y),\text{ where } F_n^{(\alpha)}(x,y)=L_n^{(\alpha)}(x)\,y^n.$$

The corresponding extended form of the group generated by R_1 is given by

$$(2.4)\quad (\exp wR_1)f(x,y)=(1-w)^{-\alpha-1}\exp\left(\frac{-wxy}{1-wy}\right)f\left(\frac{x}{1-wy},\ \frac{y}{1-wy}\right).$$

Also for the Gegenbauer polynomials $C_n^{\lambda}(x)$ defined by

$$(2.5)\quad (1-2zt+t^2)^{-\lambda}=\sum_{n=0}^{\infty} C_n^{\lambda}(z)\, t^n,$$

we consider the operator R_2, where

$$(2.6)\qquad R_2=(z^2-1)t\frac{\partial}{\partial z}+zt^2\frac{\partial}{\partial t}+2\lambda zt$$

such that

$$(2.7)\qquad R_2[F_n^{\lambda}(z,t)]=(n+1)F_{n+1}^{\lambda}(z,t),\ \text{ where }\ F_n^{\lambda}(z,t)=C_n^{\lambda}(z)t^n$$

The corresponding extended form of the group generated by R_2 is given by

$$(2.8)\qquad (\exp wR_2)\, f(z,t)$$

$$=(w^2t^2-2wzt+1)^{-\lambda} f\left(\frac{z-wt}{\sqrt{w^2t^2-2wzt+1}},\ \frac{t}{\sqrt{w^2t^2-2wzt+1}}\right)$$

Let us consider the bilateral generating function

$$(2.9)\qquad G(x,z,w)=\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x) C_n^{\lambda}(z) w^n$$

Replacing w by $wytv$, we get

$$(2.10)\qquad G(x, z, wytv)=\sum_{n=0}^{\infty} a_n\{L_n^{(\alpha)}(x)y^n\}\{C_n^{\lambda}(z)t^n\}(wv)^n$$

$$=\sum_{n=0}^{\infty} a_n F_n^{(\alpha)}(x, y)F_n^{\lambda}(z, t)(wv)^n$$

Applying the operator $(\exp wR_1)(\exp wR_2)$ on both sides of (2.10) we get

$$(2.11)\qquad (\exp wR_1)(\exp wR_2), G(x, z, wytv)$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty} a_n \frac{w^m}{m!} R_1^m F_n^{(\alpha)}(x, y) \frac{w^p}{p!} R_2^p F_n^{\lambda}(z, t)(wv)^n$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\min(m,n)} a_{n-p}(n-p+1)_{m-p}(n-p+1)_p v^{n-p} \frac{w^{n+m-p}}{(m-p)!\,p!} F_{n+m-2p}^{(\alpha)}(x, y)\, F_n^{\lambda}(z, t)$$

where $(a)_n=a(a+1)\ldots\ldots\ldots(a+n-1)$.

But

$$(\exp wR_1)(\exp wR_2)\, G(x, z, wytv)$$

$$(2.12)\qquad =(1-w)^{-\alpha-1}(w^2t^2-2wzt+1)^{-\lambda} \exp\left(\frac{-wxy}{1-wy}\right).$$

$$.G\left(\frac{x}{1-wy}, \frac{z-wt}{\sqrt{w^2t^2-2wzt+1}}, \frac{wytv}{(1-wy)\sqrt{w^2t^2-2wzt+1}}\right).$$

So we get from (2.11) and (2.12)

$$(1-w)^{-\alpha-1}(w^2-2wz+1)^{-\lambda} \exp\left(-\frac{wx}{1-w}\right).$$

$$.G\left(\frac{x}{1-w}, \frac{z-w}{\sqrt{w^2-2wz+1}}, \frac{wv}{(1-w)\sqrt{w^2-2wz+1}}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f_{m,n}(w, v, x)\; C_n^{\lambda}(z)$$

where $f_{m,n}(w,v,x)=\sum_{p=0}^{\min(m,n)}\binom{m+n-2p}{n-p}\binom{n}{p} a_{n-p} w^{m+n-p} v^{n-p} L_{n+m-2p}^{(\alpha)}(x)$,

on putting $y=t=1$, which is (1.6).

3. Application : As an application of the above theorem we consider the following generating function due to Weisner :

$$(1-2wz+w^2)^{-\lambda}\exp\left\{-\frac{wx(z-w)}{1-2wz+w^2}\right\}\ {}_0F_1\left[-\ ;\ \lambda+\tfrac{1}{2}\ ;\ \frac{x^2w^2(z^2-1)}{4(1-2wz+w^2)^2}\right]$$

$$(3.1)\qquad =\sum_{n=0}^{\infty}\frac{n\,!}{(2\lambda)_n}\,L_n^{(2\lambda-1)}(x)\ C_n^{\lambda}(z)w^n.$$

Putting $a_n=\dfrac{n\,!}{(2\lambda)_n}$, $\alpha=2\lambda-1$, $v=1$ in our theorem, we obtain

$$\rho^{-2\lambda}\exp\left[-\frac{wx}{1-w}\left\{1+\frac{(1-w)(z-w)-w}{\rho^2}\right\}\right]{}_0F_1\left[-\ ;\ \lambda+\tfrac{1}{2}:\ \frac{x^2w^2(z^2-1)}{4\rho^4}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f_{m,n}(w,x)\ C_n^{\lambda}(z),$$

where $f_{m,n}(w,x)=\displaystyle\sum_{p=0}^{\min(m,n)}\binom{n+m-2p}{n-p}\binom{n}{p}\frac{(n-p)\,!}{(2\lambda)_{n-p}}\,w^{m+n-p}L_{n+m-2p}^{(2\lambda-1)}(x)$,

and $\rho^2=(1-w)^2(w^2-2wz+1)-2w(z-w)(1-w)+w^2$,

which (viz, this particular case) also does not seem to appear earlier.

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