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BIANCHI AND VEBLEN IDENTITIES FOR THE PROJECTIVE CURVATURE TENSOR OF A SEMI-SYMMETRIC AFFINE CONNECTION IN AN AFFINELY CONNECTED SPACE

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0. Introduction.

Let V_N be an N-dimensional affinely connected space with a symmetric affine connection Γ^i_{jk}. An affine connection L^i_{jk} given by

$$(0.1)\quad L^i_{jk} = \Gamma^i_{jk} + \delta^i_j\,\phi_k - \delta^i_k\,\phi_j$$

where ϕ_j is a covariant vector, is called a semi-symmetric affine connection [1, p.36] in V_N. If B^i_{jkl} and L^i_{jkl} are the curvature tensors with respect to Γ^i_{jk} and L^i_{jk} respectively, then

$$(0.2)\quad L^i_{jkl} = B^i_{jkl} + \delta^i_k\,\phi_{jl} - \delta^i_l\,\phi_{jk} + \delta^i_j\,(\phi_{lk} - \phi_{kl})$$

where

$$(0.3)\quad \varphi_{jk} = \nabla_k\phi_j + \phi_j\phi_k = \overline{\nabla}_k\phi_j + \phi_j\phi_k,$$

∇ and $\overline{\nabla}$ being the operators of covariant differentiation with respect to the connections Γ^i_{jk} and L^i_{jk} respectively.

The projective curvature tensors for the connection Γ^i_{jk} and L^i_{jk} are given by

$$(0.4)\quad W^i_{jkl} = B^i_{jkl} + \frac{2}{N+1}\,\delta^i_j\,\beta_{kl} + \frac{1}{N-1}\left(\delta^i_k\,B_{jl} - \delta^i_l\,B_{jk}\right) + \frac{2}{N^2-1}\left(\delta^i_l\,\beta_{jk} - \delta^i_k\,\beta_{jl}\right)$$

and

$$(0.5)\quad P^i_{jkl} = L^i_{jkl} + \frac{2}{N+1}\,\delta^i_j\,\lambda_{kl} + \frac{1}{N-1}\left(\delta^i_k\,L_{jl} - \delta^i_l\,L_{jk}\right) + \frac{2}{N^2-1}\left(\delta^i_l\,\lambda_{jl} - \delta^i_k\,\lambda_{jl}\right)$$

where $B_{jk} = B^t_{jkt}$, $L_{jk} = L^t_{jkt}$

$2\beta_{jk} = B_{jk} - B_{kj}$, $2\lambda_{jk} = L_{jk} - L_{kj}$

It is known that the curvature tensor of any symmetric connection satisfies the Bianchi identity and the Veblen identity [1, p.56] which are

$$(0.6) \quad \nabla_m B^i_{jkl} + \nabla_k B^i_{jlm} + \nabla_l B^i_{jmk} = 0$$

and

$$(0.7) \quad \nabla_m B^i_{jkl} + \nabla_j B^i_{mlk} + \nabla_k B^i_{ljm} + \nabla_l B^i_{kmj} = 0$$

The Bianchi identity [2] and Veblen identity [3] for the projective curvature tensor in a Riemannian space are

$$(0.8) \quad \nabla_m W^i_{jkl} + \nabla_k W^i_{jlm} + \nabla_l W^i_{jmk} - \frac{1}{N-2} \nabla_t \left(\delta^i_m W^t_{jkl} + \delta^i_k W^t_{jlm} + \delta^i_l W^t_{jmk} \right) = 0$$

and

$$(0.9) \quad \nabla_m W^i_{jkl} + \nabla_j W^i_{mlk} + \nabla_k W^i_{ljm} + \nabla_l W^i_{kmj} - \frac{1}{N-2} \nabla_t \left(\delta^i_m W^t_{jkl} + \delta^i_j W^t_{mlk} + \delta^i_k W^t_{ljm} + \delta^i_j W^t_{kmj} \right) = 0$$

In this paper analogous identities for the projective curvature tensor P^i_{jkl} of a semi-symmetric affine connection in V_N have been derived.

1. Bianchi and Veblen identities for the curvature tensor L^i_{jkl}.

From (0, 2), we get

$$(1.1) \quad L^i_{jkl} + L^i_{klj} + L^i_{ljk} = 2\left[\delta^i_j (\phi_{lk} - \phi_{kl}) + \delta^i_k (\phi_{jl} - \phi_{lj}) + \delta^i_l (\phi_{kj} - \phi_{jk})\right].$$

Therefore, $L^i_{jkl} + L^i_{klj} + L^i_{ljk} = 0$ iff $\phi_{jk} = \phi_{kj}$ for $N \geqslant 3$. But from (0.3), this condition is equivalent to $\nabla_j \phi_k = \nabla_k \phi_j$ which implies that ϕ_i is a gradient vector. Thus

Theorem 1. The curvature tensor of a semi-symmetric affine connection satisfies

$$L^i_{jkl} + L^i_{klj} + L^i_{ljk} = 0 \text{ iff } \phi_j \text{ is a gradient vector.}$$

In the remaining part of this paper ϕ_j will be considered as a gradient vector.

Let w_i be an arbitrary non-null covariant vector in V_N. The generalized Ricci identity gives

$$\overline{\nabla}_k\overline{\nabla}_j w_i - \overline{\nabla}_j\overline{\nabla}_k w_i = w_t L^t_{ijk} - 2\{(\overline{\nabla}_j w_i)\phi_k - (\overline{\nabla}_k w_i)\phi_j\} \tag{1.2}$$

Operating both sides by $\overline{\nabla}_l$ we get

$$\overline{\nabla}_l\overline{\nabla}_k\overline{\nabla}_j w_i - \overline{\nabla}_l\overline{\nabla}_j\overline{\nabla}_k w_i = (\overline{\nabla}_l w_t) L^t_{ijk} + w_t\overline{\nabla}_l L^t_{ijk}$$
$$- 2\{(\overline{\nabla}_l\overline{\nabla}_j w_i)\phi_k - (\overline{\nabla}_l\overline{\nabla}_k w_i)\phi_j + (\overline{\nabla}_j w_i)(\overline{\nabla}_l\phi_k) - (\overline{\nabla}_k w_i)(\overline{\nabla}_l\phi_j)\} \tag{1.3}$$

Permuting j, k, l cyclically and then adding all possible expressions obtained from (1.3), we get, by virtue of (1.2)

$$\overline{\nabla}_l(\overline{\nabla}_k\overline{\nabla}_j w_i - \overline{\nabla}_j\overline{\nabla}_k w_i) + \overline{\nabla}_k(\overline{\nabla}_j\overline{\nabla}_l w_i - \overline{\nabla}_l\overline{\nabla}_j w_i)$$
$$+ \overline{\nabla}_j(\overline{\nabla}_l\overline{\nabla}_k w_i - \overline{\nabla}_k\overline{\nabla}_l w_i) \tag{1.4}$$
$$= \{(\overline{\nabla}_l w_t) L^t_{ijk} + (\overline{\nabla}_k w_t) L^t_{ilj} + (\overline{\nabla}_j w_t) L^t_{ikl}\}$$
$$+ w_t(\overline{\nabla}_l L^t_{ijk} + \overline{\nabla}_j L^t_{ikl} + \overline{\nabla}_k L^t_{ilj}) + 2w_t(\phi_l L^t_{ijk} + \phi_j L^t_{ikl} + \phi_k L^t_{ilj})$$

Applying generalized Ricci identity to $\overline{\nabla}_j w_i$ we get

$$\overline{\nabla}_l\overline{\nabla}_k(\overline{\nabla}_j w_i) - \overline{\nabla}_k\overline{\nabla}_l(\overline{\nabla}_j w_i) = (\overline{\nabla}_t w_i) L^t_{jkl} + (\overline{\nabla}_j w_t) L^t_{ikl}$$
$$- 2\{(\overline{\nabla}_k\overline{\nabla}_j w_i)\phi_l - (\overline{\nabla}_l\overline{\nabla}_j w_i)\phi_k\} \tag{1.5}$$

Adding the expressions obtained from (1.5) by all possible cyclic permutations of j, k, l and using the resulting equation in (1.4), we get, by virtue of Theorem 1,

$$2\{(\overline{\nabla}_l w_t) L^t_{ijk} + (\overline{\nabla}_k w_t) L^t_{ilj} + (\overline{\nabla}_j w_t) L^t_{ikl}\}$$
$$+2w_t(\overline{\nabla}_l L^t_{ijk} + \overline{\nabla}_j L^t_{ikl} + \overline{\nabla}_k L^t_{ilj})$$
$$+4w_t(\phi_l L^t_{ijk} + \phi_j L^t_{ikl} + \phi_k L^t_{ilj})$$
$$=2\{(\overline{\nabla}_l w_t) L^t_{ijk} + (\overline{\nabla}_j w_t) L^t_{ikl} + (\overline{\nabla}_k w_t) L^t_{ilj}\}$$
$$+w_t(\overline{\nabla}_l L^t_{ijk} + \overline{\nabla}_j L^t_{ikl} + \overline{\nabla}_k L^t_{ilj})$$

From the above equation we get

$$(\overline{\nabla}_l L^t_{ijk} + 4\phi_l L^t_{ijk}) + (\overline{\nabla}_j L^t_{ikl} + 4\phi_j L^t_{ikl})$$
$$+ (\overline{\nabla}_k L^t_{ilj} + 4\phi_k L^t_{ilj}) = 0 \tag{1.6}$$

Put

(1.7) $\bar{L}^t_{ijk} = e^{4\phi} L^t_{ijk}$, $\bar{L}_{ij} = e^{4\phi} L_{ij}$, $\bar{\lambda}_{ij} = e^{4\phi}\lambda_{ij}$

Then

(1.8) $\bar{\nabla}_l \bar{L}^t_{ijk} = e^{4\phi}(\bar{\nabla}_l L^t_{ijk} + 4\phi_l L^t_{ijk})$, $\bar{\nabla}_k \bar{L}_{ij} = e^{4\phi}(\bar{\nabla}_k L_{ij} + 4\phi_k L_{ij})$

Substituting from (1.8) in (1.6), we get

(1.9) $\bar{\nabla}_l \bar{L}^t_{ijk} + \bar{\nabla}_j \bar{L}^t_{ikl} + \bar{\nabla}_k \bar{L}^t_{ilj} = 0$

This is the Bianchi identity for the curvature tensor $\bar{L}^t_{ijk}$ of the semi-symmetric affine connection (0.1), for which ϕ_j is a gradient vector.

From Theorem 1, and (1.7) we get

$$\bar{L}^t_{ijk} + \bar{L}^t_{jki} + \bar{L}^t_{kij} = 0.$$

Applying $\bar{\nabla}_l$ we get

(1.10) $\bar{\nabla}_l \bar{L}^t_{ijk} = \bar{\nabla}_l \bar{L}^t_{jik} + \bar{\nabla}_l \bar{L}^t_{kji}$.

Similarly,

(1.11) $\bar{\nabla}_j \bar{L}^t_{ikl} = \bar{\nabla}_j \bar{L}^t_{kil} + \bar{\nabla}_j \bar{L}^t_{lki}$

Applying (1.10) and (1.11) in (1.9) and using (1.9) again in the resulting equation, we get

(1.12) $\bar{\nabla}_l \bar{L}^t_{jik} + \bar{\nabla}_j \bar{L}^t_{lki} + \bar{\nabla}_i \bar{L}^t_{kjl} + \bar{\nabla}_k \bar{L}^t_{ilj} = 0.$

This is the Veblen identity for the curvature tensor $\bar{L}^t_{jkl}$ of the semi-symmetric affine connection (0.1) for which ϕ_j is a gradient vector.

2. Bianchi and Veblen identities for the projective curvature tensor $\bar{P}^t_{jkl}$:

From (0.5) and (1.7) we can write

(2.1) $\bar{P}^t_{ijk} = e^{4\phi} P^t_{ijk}$

By virtue of (2.1) and (1.9) we have

(2.2) $\bar{\nabla}_l \bar{P}^t_{ijk} + \bar{\nabla}_j \bar{P}^t_{ikl} + \bar{\nabla}_k \bar{P}^t_{ilj} = \frac{2}{N+1}\delta^t_i(\bar{\nabla}_j \bar{\lambda}_{kl} + \bar{\nabla}_k \bar{\lambda}_{lj} + \bar{\nabla}_l \bar{\lambda}_{jk})$

$$+ \frac{1}{N^2-1}[\delta^t_l \{\bar{\nabla}_k(N\bar{L}_{ij} + \bar{L}_{ji}) - \bar{\nabla}_j(N\bar{L}_{ik} + \bar{L}_{ki})\}$$

$$+ \delta^t_k \{\bar{\nabla}_j(N\bar{L}_{il} + \bar{L}_{li}) - \bar{\nabla}_l(N\bar{L}_{ij} + \bar{L}_{ji})\}$$

$$+ \delta^t_j \{\bar{\nabla}_l(N\bar{L}_{ik} + \bar{L}_{ki}) - \bar{\nabla}_k(N\bar{L}_{il} + \bar{L}_{li})\}]$$

Contracting t and l in (1.9) and applying the result to

$L^t_{ijk} + L^t_{jki} + L^t_{kij} = 0$, we get

$$(2.3)\quad \overline{\nabla}_j\lambda_{kl} + \overline{\nabla}_k\bar{\lambda}_{ij} + \overline{\nabla}_l\bar{\lambda}_{jk} = 0.$$

Again, contracting t and l in (2.2), we get, by virtue of (2.3),

$$(2.4)\quad \overline{\nabla}_h\bar{P}^h_{ijk} = \frac{N-2}{N-1}[\overline{\nabla}_k(N\bar{L}_{ij} + \bar{L}_{ji}) - \overline{\nabla}_j(N\bar{L}_{ik} + \bar{L}_{ki})].$$

Applping (2.4) to (2.2), we get

$$(2.5)\quad \overline{\nabla}_l\bar{P}^t_{ijk} + \overline{\nabla}_j\bar{P}^t_{ikl} + \overline{\nabla}_k\bar{P}^t_{ilj} - \frac{1}{N-2}\overline{\nabla}_h(\delta^t_l\,\bar{P}^h_{ijk} + \delta^t_j\,\bar{P}^h_{ikl} + \delta^t_k\,\bar{P}^h_{ilj}) = 0.$$

This is the Bianchi identity for the projective curvature tensor of a semi-symmetric affine connection for which ϕ_j is a gradient vector.

The Veblen identity for the projective curvature tensor of a semi-symmetric affine connection for which ϕ_j is a gradient vector, is given by

$$(2.6)\quad \overline{\nabla}_l\bar{P}^t_{jik} + \overline{\nabla}_j\bar{P}^t_{lki} + \overline{\nabla}_i\bar{P}^t_{kjl} + \overline{\nabla}_k\bar{P}^t_{ilj} - \frac{1}{N-2}\overline{\nabla}_h(\delta^t_l\,\bar{P}^h_{jik} + \delta^t_j\,\bar{P}^h_{lki} + \delta^t_i\,\bar{P}^h_{kjl} + \delta^t_k\,\bar{P}^h_{ilj}) = 0.$$

The calculation is same as for the Veblen identity (1.12) for the tensor $\bar{L}^t_{ijk}$

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