Jour. Pure Math.
Vol. I (1981), pp. 61-66

KINNEY'S FUNCTIONS IN STUDY OF SOME PROPERTIES OF THE CANTOR SET

D. K. GANGULY

1. Introduction and Notations :

For any $x \in [0, 1]$, let $x = \sum_{i=1}^{\infty} \frac{x_i}{3^i}$, $x_i = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ for every i.

Kinney [3] defined two functions $f(x)$ and $\nu(x)$ by the relations :

$$f(x) = \sum_{i=1}^{\infty} \frac{f_i(x)}{3^i} \text{ and } \nu(x) = \sum_{i=1}^{\infty} \frac{\nu_i(x)}{3^i} \text{ where } f_i(x) = 2\delta(x_i, 2) \text{ and } \nu_i(x) = 2\delta(x_i, 1)$$

with the properties that $\delta(a, b) = 1$ if $a = b$ and $\delta(a, b) = 0$ if $a \neq b$. It follows that $x = f(x) + \frac{1}{2}\nu(x)$ and $f(x) \in c$ and $\nu(x) \in C$ where C is the Cantor set for any $x \in [0, 1]$.

To avoid the ambiguity, we make the convention that any ternary representation of x in [0. 1] should not end with a chain of 2's but we take $f(1) = 1$ and $\nu(1) = 0$, by definitions.

It is known that the Cantor set C possesses the Steinhaus property as well as the Randolph property i.e. for any $d \in [0, 1]$ we can find a pair x_1, x_2 of Cantor points such that $x_2 - x_1 = d$ and also a pair y_1, y_2 of Cantor points such that $\frac{y_1 + y_2}{2} = d$, ([1], [4], [5], [7]).

A set E is said to be an (SD) – set if its distance set fills an interval about the origin, whose length is equal to the diameter of the set E [2].

A set E defined in [0, 1] is said to be symmetrical if x and $(1 - x)$ both belong to E.

Cantor set C is an (SD) – set and is also symmetrical. That the Cantor set C possesses both the Steinhaus and the Randolph properties has been shown by the above mathematicians.

I propose to give here another method of the proof of the same properties, which seems to me to be shorter and direct, compared to the proofs given earlier.

Theorem (Randolph). Every point between 0 and 1 is the arithmatical mean of at least one pair of Cantor points.

Proof: Let $f(x)$ and $v(x)$ be the Kinney's functions defined in [0, 1]. If x is any point in [0, 1], then $x=f(x)+\frac{1}{2}v(x)$ ... ... (1)

and $x=\sum_{i=1}^{\infty}\frac{x_i}{3^i}$, $x_i=\begin{pmatrix}0\\1\\2\end{pmatrix}$ for every $i=1, 2, 3,$.........

It follows that, $f(x)=\sum_{i=1}^{\infty}\frac{f_i(x)}{3^i}$ and $v(x)=\sum_{i=1}^{\infty}\frac{v_i(x)}{3^i}$

where $f_i(x)=2\delta(0, 2)=0$ and $v_i(x)=2\delta(0, 1)=0$,

when $x_i=0$.

Thus $f_i(x)+v_i(x)=0$, when $x_i=0$,

Again $f_i(x)=2\delta(1, 2)=0$ and $v_i(x)=2\delta(1, 1)=2$,

when $x_i=1$.

Thus $f(x)+v_i(x)=2$, when $x_i=1$.

Finally $f_i(x)=2\delta(2, 2)=2$ and $v_i(x)=2\delta(2, 1)=0$,

when $x_i=2$.

Thus $f_i(x)+v_i(x)=2$, when $x_i=2$.

Therefore, $f(x)+v(x)=\sum_{i=1}^{\infty}\frac{f_i(x)+v_i(x)}{3^i}=\sum_{i=1}^{\infty}\frac{a_i}{3^i}$

where $a_i=f_i(x)+v_i(x)=0$ or 2 for every $i=1, 2, 3$.........

It follows that $\psi(x)=f(x)+v(x)\in C$ for every $x\in[0, 1]$.

Now from (1), we have $2x=2f(x)+v(x)=f(x)+\{f(x)+v(x)\}=f(x)+\psi(x)$.

Thus $x=\frac{f(x)+\psi(x)}{2}$ where $f(x)\in C$, $\psi(x)\in C$ and x is any point in [0, 1]. Hence the theorem.

Corollary (Steinhau's Theorem): The distance set of the Cantor middle third set C fills the unit interval $0\leqslant d\leqslant 1$ [i.e. C is an (SD) – set].

Proof: The proof of this theorem follows from Randolph's theorem and a theorem given by Bose Majumdar [1] stated below.

"A necessary and sufficient condition that a linear symmetrical set E defined in [0, 1] may be an (SD) – set is that it satisfies Randolph's property".

[In fact, if $d \in [0, 1]$, then taking $1-2x=d$, we get $0 < x < \frac{1}{2}$ and hence by Randolph's theorem $x=\frac{f(x)+\varphi(x)}{2}$ where $f(x) \in C$ and $\psi(x) \in C$ and thus $d=f_1(x)-\psi(x)$ where $f_1(x)=1-f(x) \in C$].

2. In elementary books on algebra and trigonometry methods are given for solving equations and also constructing equations when roots are given.

In the following particular case, we propose to show that we can construct equations whose roots are precisely all the numbers of the perfect set C and no number outside it.

Theorem : Each of the equations (1) $f(x)=x$ and (2) $v(x)=0$ are satisfied if and only if $x \in C$, where $f(x)$ and $v(x)$ are Kinney's functions defined in $[0, 1]$, and C is the Cantor set.

Proof : The condition is sufficient.

We are given any $x \in C$. It is to be shown that x is a solution of each of the equation (1) and (2).

Let $x=\sum_{i=1}^{\infty}\frac{a_i}{3^i}$ where $a_i=\begin{pmatrix}0\\2\end{pmatrix}$ for each $i=1, 2, 3......$

Then $f(x)=\sum_{i=1}^{\infty}\frac{f_i(x)}{3^i}$ and $v(x)=\sum_{i=1}^{\infty}\frac{v_i(x)}{3^i}$

Now, $f_i(x)=0$ and $v_i(x)=0$, when $a_i=0$

Again $f_i(x)=2$ and $v_i(x)=0$, when $a_i=2$.

Hence $f_i(x)=a_i$ and $v_i(x)=0$ for every $i=1, 2, 3......$

Thus $f(x)=\sum_{i=1}^{\infty}\frac{f_i(x)}{3^i}=\sum_{i=1}^{\infty}\frac{a_i}{3^i}=x$

and $$v(x)=\sum_{i=1}^{\infty}\frac{v_i(x)}{3^i}=\sum_{i=1}^{\infty}\frac{0}{3^i}=0.$$

Hence every $x \in C$ satisfies the equations (1) and (2).

The condition is necessary.

Let $x \in [0, 1]$ be such that $f(x)=x$ and $v(x)=0$.

Then we show that x is necessarily a Cantor point.

Let $f(x)=\sum_{i=1}^{\infty}\frac{f_i(x)}{3^i},\ v(x)=\sum_{i=1}^{\infty}\frac{v_i(x)}{3^i},$

where $x=\sum_{i=1}^{\infty}\frac{\alpha_i}{3^i},\ \alpha_i=\begin{pmatrix}0\\1\\2\end{pmatrix}$ for every $i=1, 2, 3$.........

Now, if $\alpha_i=0$, $f_i(x)=0$ and $v_i(x)=0$

if $\alpha_i=1$, $f_i(x)=0$ and $v_i(x)=2$

if $\alpha_i=2$, $f_i(x)=2$ and $v_i(x)=0$.

Thus $f_i(x)=\alpha_i$ if and only if $\alpha_i=0$ or 2.

and $v_i(x)=0$ if and only if $\alpha_i=0$ or 2.

It follows that $f(x)=x$ implies that $\sum_{i=1}^{\infty}\frac{f_i(x)}{3^i}=\sum_{i=1}^{\infty}\frac{\alpha_i}{3^i}$

which implies that $\alpha_i=f_i(x)$ for all positive integers i and this happens only when $\alpha_i=0$ or 2 (and never 1) and this implies $x\in C$.

Therefore $f(x)=x$ implies $x\in C$. Similarly $v(x)=0$ implies that $\sum_{i=1}^{\infty}\frac{v_i(x)}{3^i}=0$ which implies that $v_i(x)=0$ for all $i=1, 2, 3,$.........and this happens only when $\alpha_i=0$ or 2 (and never 1) and this implies that $x\in c$.

Hence the theorem is completely proved.

Corollary : Kinney's function $f(x)$ has the following property on the set C, viz $f\left(\frac{x_1+x_2}{2}\right)<\frac{f(x_1)+f(x_2)}{2}$ for any two points x_1 and x_2 belonging to C (This property of $f(x)$ is some what similar to that of functions convex downward).

Proof; Let x_1 and x_2 be any two points of C.

Then $f(x_1)=x_1=\sum_{i=1}^{\infty}\frac{a_i}{3^i},\ a_i=\begin{pmatrix}0\\2\end{pmatrix}$ for every $i=1, 2, 3$......

and $f(x_2)=x_2=\sum_{i=1}^{\infty}\frac{b_i}{3^i},\ b_i=\begin{pmatrix}0\\2\end{pmatrix}$ for every $i=1, 2, 3$......

Hence

(1) $\dfrac{f(x_1)+f(x_2)}{2}=\dfrac{x_1+x_2}{2}=\sum_{i=1}^{\infty}\dfrac{\frac{a_i+b_i}{2}}{3^i}=\sum_{i=1}^{\infty}\dfrac{c_i}{3^i}$

We now write $x=\dfrac{x_1+x_2}{2}$ where $x_1 \in C$ and $x_2 \in C$.

Then

(2) $f\left(\dfrac{x_1+x_2}{2}\right)=f(x)=\sum_{i=1}^{\infty}\dfrac{f_i(x)}{3^i}$.

where $x=\sum_{i=1}^{\infty}\dfrac{c_i}{3^i}$, by (1)

If $\begin{cases} a_i=0 \\ b_i=0, \end{cases}$ $c_i=0$ and thus $f_i(x)=0=c_i$

If $\begin{cases} a_i=0 \\ b_i=2, \end{cases}$ $c_i=1$ and thus $f_i(x)=0 < c_i$

If $\begin{cases} a_i=2 \\ b_i=0, \end{cases}$ $c_i=1$ and thus $f_i(x)=0 < c_i$

If $\begin{cases} a_i=2 \\ b_i=2, \end{cases}$ $c_i=2$ and thus $f_i(x)=2=c_i$

In any case, therefore $f_i(x) \leqslant c_i$ for every $i=1, 2, 3......$

Hence for (2) $f\left(\dfrac{x_1+x_2}{2}\right)=\sum_{i=1}^{\infty}\dfrac{f_i(x)}{3^i}=\dfrac{f(x_1)+f(x_2)}{2}$ by (1)

This proves the theorem.

Note: We know that any point $x \in C$ is the middle point of a unique pair of points x_1, x_2 where $x_1 \in C$, $x_2 \in C$ [6]

That is, $\dfrac{x_1+x_2}{2}=x$ where $x \in c$, $x_1 \in c$, $x_2 \in c$.

Hence by above corollary $f(x)=f\left(\dfrac{x_1+x_2}{2}\right) \leqslant \dfrac{f(x_1)+f(x_2)}{2}$ where x_1, x_2 and $\dfrac{x_1+x_2}{2}$ are all points of C.

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Received
6.7.1981

Dept. of Pure Math.
Calcutta University