Jour. Pure Math.
Vol. I (1981), pp. 39-44

GENERAL REAL INVERSION OPERATOR

D. K. BHATTACHARYA

Abstract : The paper deals with the inversion formula of Laplace-Stieltjes integrals in one and more variables by using a general real inversion operator from which similar operators used by A. Erdelyi [2] and P. G. Rooney [3] follow as particular cases.

1.1. Introduction :

$$\text{Let } f(s) = \int_0^\infty e^{-st}\, d\{\alpha(t)\}$$

then the real inversion operator is taken as

$$\int_0^t L_{k,\,t}^{\lambda,\,\mu,\,\nu,} [\,f(s)\,] \quad dt, \text{ where}$$

$$L_{k,\,t}^{\lambda,\,\mu,\,\nu,} [\,f(s)] = (A_1) \int_0^\infty x^\nu\, [J_{kx}]_\nu^\mu\; f\left[\frac{k(x+1)}{t}\right] dx,\ \lambda \geqslant 0,\ \ \nu > -1,\ (\mu,\, k) > 0.$$

$$\text{and} \quad A_1 = k^{\lambda+\nu+\frac{3}{2}}\, 2^{2\lambda-\frac{1}{2}}\, \{\mu(\mu+1)^{\frac{1}{2}}\}\, (t\ \sqrt{\pi})^{-1}\, [W_{\lambda,\nu}\, (4k)^{-1}]$$

$$[J_{kx}]_\nu^\mu = J_\nu^\mu\, (k^{\mu+1}\, x^\mu),$$

$$J_\nu^\mu\, (x) = \sum_{k=0}^\infty \frac{(-x)^k}{k\,!\,\Gamma(1+\nu+\mu k)}$$

$$\text{Since} \quad J_\nu^1\, (x) = x^{-\frac{\nu}{2}}\, J_\nu\, (2x^{\frac{1}{2}})$$

$$\text{and} \quad W_{0,\nu}\, (x) = \left(\frac{x}{\pi}\right)^{\frac{1}{2}} K_\nu \left(\frac{x}{2}\right).$$

we have,

$$L_{k,t}^{0,1,\nu} [\,f(s)] = [2t\, K_\nu\, (2k)]^{-1}\, k \int_0^\infty x^{\frac{\nu}{2}}\, J_\nu\, (2K\, x^{\frac{1}{2}})\, f\left[\frac{k(x+1)}{t}\right] dx$$

—the operator $L_{k,t}\ [\,f(s)]$ discussed by A. Erdelyi [2].

Again $L_{k,t}^{0,1,\frac{1}{2}}\ [\,f(s)]$ are the cases discussed by P. G. Rooney [3].

Next Let $f(s.t) = \int_0^\infty \int_0^\infty e^{-sx-ty} \, d\{\alpha(x, y)\}$, then the real inversion operator is of the form

$$\int_0^x \int_0^y L^{\lambda;\mu_1,\mu_2;\nu_1,\nu_2}_{k_1,k_2;x,y} [f(s, t)] \, dx \, dy$$

$$=(A_2)\int_0^\infty \int_0^\infty \prod_1^2 \left[z_i^{\nu_i} \left\{ J_{kz} \right\}_{\nu_i}^{\mu_i} \right] f\left[\frac{k_1(z_1+1)}{x}, \frac{k_2(z_2+1)}{y} \right] dz_1 \, dz_2,$$

$\lambda \geqslant 0$, $\nu_i > -1$, $(\mu_i, k_i) > 0$, $i = 1, 2$

where $A_2 = \prod_1^2 \left[k_i^{\lambda+\nu_i+\frac{3}{2}} \, 2^{4\lambda-1} \{\mu_i(\mu_i+1)\}^{\frac{1}{2}} \, (\pi xy)^{-1} \, \{W_{\lambda_i,\nu_i}(4K_i)\}^{-1} \right]$

1.2. Notations.

In what follows, we use the following notations.

(a) A function $\phi(x, y)$ is said to be quasi-normalized on $S : [o, o\,; R_1, R_2]$ if $\phi(x, y) = \frac{1}{4}[\phi]_{x\pm,\, y\pm}$,

where, $[\phi]_{x\pm\, y\pm} = \phi(x+, y+) + \phi(x+, y-) + \phi(x-, y+) + \phi(x-, y-)$.

and $\phi(x\pm, y\pm)$ means the limit of $\phi(x, y)$ as $(x, y) \to (x\pm, y\pm)$.

along the line joining the points (x, y) and $(x\pm, y\pm)$.

(b) A function $\phi(x, y)$ belongs to the class H on S, if

(i) for all possible rectangles consisting of lines of the form

$x = x_i$, $y = y_j$ $(i = 1, 2, \ldots, m\,; j = 1, 2, \ldots, n)$

the set of all summations

$$\sum_{i,j} | \{\phi\}_{D_{ij}} |, \; D_{i,j} = [x_{i-1}, y_{j-1}\,; x_i, y_j]$$

is bounded on S.

(ii) $\phi(x_0, y)$ and $\phi(x, y_0)$ for some fixed values x_0, y_0 are also functions of bounded variation on S,

(c) A function $\phi(x, y)$ belongs to the class H_0 on S. if

(i) $\phi \in$ H on S

(ii) $\phi(x, 0) = 0 = \phi(0, y)$.

1.3 Theorem 1.3.1.

Let (i) $\alpha(t)$ be a normalized function of bounded variation in $(0, R)$, $R > 0$,

(ii) $\int_0^\infty e^{-st}\, d\{\alpha(t)\}$

converges with $s=\nu > 0$, then

$$\lim_{k\to\infty} \int_0^t L_{k,t}^{\lambda,\mu,\nu}[f(s)]\, dt = \tfrac{1}{2}[\alpha(t+)+\alpha(t-)]$$

almost everywhere for $t > 0$, where $\alpha(t\pm)$ exist.

Proof : By hypothesis,

$$f(s)=s.\int_0^\infty e^{-st}\,\alpha(t)\, dt.$$

Now $$L_{k,t}^{\lambda,\mu,\nu}\left[f(s)\right]=(A_1).\int_0^\infty x^\nu \left[J_{kx}\right]_\nu^\mu \left\{\int_0^\infty \frac{k(x+1)}{t}\, e^{\frac{-k(x+1)u}{t}}\alpha(u)\, du\right\}$$

$$=\left(\frac{k}{t}\right)(A_1).\int_0^\infty e^{\frac{-ku}{t}}\alpha(u)\left\{\int_0^\infty e^{\frac{-kxu}{t}}\left(x^{\nu+1}+x^\nu\right)\left[J_{kx}\right]_\nu^\mu dx\right\}du$$

[by change of order of integration, which is easily justifiable]

$$=\left(\frac{k}{t}\right).(A_1).\,k^{-(\nu+2)}\,t^{\nu+2}\int_0^\infty u^{-(\nu+2)}\,e^{-k\left\{\frac{u}{t}+\left(\frac{t}{u}\right)^\mu\right\}}.\,\alpha(u)$$

$$.\left\{(\nu+1)+\left(k\frac{u}{t}\right)-\mu k\left(\frac{t}{u}\right)^\mu\right\}du.$$

$$=t.k^{-(\nu+1)}(A_1).\int_0^\infty u^{-(\nu+2)}.\,\alpha(u)\left[t^\nu\left\{(\nu+1)+\left(k\frac{u}{t}\right)-\mu k\left(\frac{t}{u}\right)^\mu\right\}.\right.$$

$$\left.e^{-k\left\{\frac{u}{t}+\left(\frac{t}{u}\right)^\mu\right\}}\right]du$$

$$=t.k^{-(\nu+1)}(A_1)\int_0^\infty u^{-(\nu+2)}.\alpha(u)\,\frac{d}{dt}\left[e^{-k\left\{\frac{u}{t}+\left(\frac{t}{u}\right)^\mu\right\}}.\,t^{(\nu+1)}\right]du$$

$$\therefore \int_0^t L_{k,t}^{\lambda,\mu,\nu}\left[f(s)\right]dt$$

$$=t.k^{-(\nu+1)}(A_1).\,t^{\nu+1}\int_0^\infty e^{-k\left\{\frac{u}{t}+\left(\frac{t}{u}\right)^\mu\right\}}u^{-(\nu+2)}.\,\alpha(u)du$$

(differentiation under the sign of integration being permissible as by hypothesis

$\int_0^\infty e^{-st} \,.\, t^{-(\nu+2)} \alpha(t)dt$ converves uniformly and absolutely for $s > \gamma$).

$$\sim \frac{k^{\lambda+\frac{1}{2}} \,.\, 2^{2\lambda-\frac{1}{2}} \{\mu(\mu+1)\}^{\frac{1}{2}}}{(\sqrt{\pi}) \,.\, e^{-2k} \,.\, (4k)^{\lambda}} \,.\, t^{\nu+1} \,.\, t^{-(\nu+2)} \,.\, \{\alpha(t+)+\alpha(t-)\} e^{-2k}$$

$$\cdot \frac{t\sqrt{\pi}}{\{2k\mu(\mu+1)\}^{\frac{1}{2}}}, \text{ as } k \to \infty.$$

(assuming that the asymptotic evaluation is justified).

$$= \frac{\alpha(t+)+\alpha(t-)}{2}, \quad t > 0.$$

Let us now justify the asymptotic evaluation.

Let I_1 and I_2 be respectively the integrals

$$k^{\lambda+\frac{1}{2}} \int_0^{t-\delta} e^{-k\left\{\frac{u}{t}+\left(\frac{t}{u}\right)^{\mu}\right\}} u^{-(\nu+2)} \alpha(u)du$$

and $$k^{\lambda+\frac{1}{2}} \int_{t+\delta}^{\infty} e^{-k\left\{\frac{u}{t}+\left(\frac{t}{u}\right)^{\mu}\right\}} u^{-(\nu+2)} \alpha(u)du$$

We choose $k_0 > \gamma t$, then for $k > k_0$,

$$|I_2| \leqslant k^{\lambda+\frac{1}{2}}\left[W_{\lambda,\mu}(4k)\right]^{-1} \int_{t+\delta}^{\infty} e^{-k_0\left\{\frac{u}{t}+\left(\frac{t}{u}\right)^{\mu}\right\}} e^{-(k-k_0)\left\{\frac{u}{t}+\left(\frac{t}{u}\right)^{\mu}\right\}}.$$

$$|u^{-(\nu+2)}\alpha(u)|\,du$$

$$< k^{\lambda+\frac{1}{2}}\left[W_{\lambda,\nu}(4k)\right]^{-1} e^{(k-k_0)}\left[\frac{(t+\delta)}{t}+\left\{\frac{(t+\delta)}{t}\right\}^{-\mu}\right].$$

$$\int_{t+}^{\infty} e^{-k_0\left\{\frac{u}{t}+\left(\frac{t}{u}\right)^{\mu}\right\}} |u^{-(\nu+2)}\alpha(u)|\,du$$

$$A(t) \,.\, k^{\lambda+\frac{1}{2}}\left[W_{\lambda,\nu}(4k)\right]^{-1} e^{-k\alpha}, \quad [\alpha > 2,\ \mu \leqslant 1]$$

$$\sim A(t) \,.\, k^{\lambda+\frac{1}{2}} e^{2k} e^{-k\alpha}$$

$$\to 0 \text{ as } k \to \infty.$$

Similarly it may be shown that $|I_1| \to 0$ as $k \to \infty$.

Thus the theorem is completely proved.

Theorem. 1.3.2

Let (i) $\alpha(x, y)$ be a normalized function belonging to H_0 on S

(ii) $\int_0^\infty \int_0^\infty e^{-(sx-ty)} d\{\alpha(x, y)\}$

converges with $s = \gamma_1 > 0$, $t = \gamma_2 > 0$,

then, $\lim_{(k_1, k_2)\to(\infty,\infty)} \int_0^x \int_0^y L^{\lambda;\mu_1,\mu_2;\nu_1,\nu_2}_{k_1,k_2;x,y} [f(s, t)]\, dx\, dy = \frac{1}{4}[\alpha]_{x\pm,\, y\pm}$

almost everywhere, where $\alpha(x\pm, y\pm)$ exist.

Proof : By hypothesis, we have

$$f(s, t) = st \int_0^\infty \int_0^\infty e^{-sx-ty} \alpha(x, y)\, dx\, dy$$

Therefore,

$$L^{\lambda;\mu_1,\mu_2.\nu_1,\nu_2}_{k_1,k_2;x,y} [f(s, t)] = (k_1 k_2/xy) \int_0^\infty \int_0^\infty e^{-k_1 \frac{u}{x}} e^{-k_2 \frac{v}{y}} \alpha(u, v).$$

$$\left[e^{-k_1 z_1 \frac{u}{x}} e^{-k_2 z_2 \frac{v}{y}} \prod_1^2 \left\{z_i^{\nu_i+1} + z_i^{\nu_i}\right\} . \left[J_{k_i z_i}\right]_{\nu_i}^{\mu_i} dz_i\right\}\Big] du\, dv$$

$$= (xy) \prod_1^2 \left\{K_i^{-(\nu_i+1)}\right\}(A_2) . \int_0^\infty \int_0^\infty u^{-(\nu_1+2)}\, v^{-(\nu_1+2)}\, \alpha(u, v).$$

$$\frac{d}{dx}\left[e^{-k_1\left\{\frac{u}{x}+\left(\frac{x}{u}\right)^{\mu_1}\right\}} x^{\nu_1+1}\right] . \frac{d}{dy}\left[e^{-k_2\left\{\frac{v}{y}+\left(\frac{y}{v}\right)^{\mu_2}\right\}} y^{\nu_2+1}\right] du\, dv$$

$$\therefore \int_0^x \int_0^y L^{\lambda;\mu_1,\mu_2;\nu_1,\nu_2}_{k_1,k_2;x,y} [f(s, t)]\, dx\, dy$$

$$= (xy)\, k_1^{-(\nu_1+1)}\, k_2^{-(\nu_2+1)} (A_2) .\, x^{\nu_1+1}\, y^{\nu_2+1}.$$

$$\left[\int_0^\infty e^{-k_1\left\{\frac{u}{x}+\left(\frac{x}{u}\right)^{\mu_1}\right\}} u^{-(\nu_1+2)} du . \int_0^\infty e^{-k_2\left\{\frac{v}{y}+\left(\frac{y}{v}\right)^{\mu_2}\right\}} v^{-(\nu_2+2)} \alpha(u, v) dv\right]$$

$$\sim k_1^{\lambda+\frac{1}{2}}\, k_2^{\lambda+\frac{1}{2}}\, 2^{4\lambda-1} \{\mu_1\mu_2(\mu_1+1)(\mu_2+1\}^{\frac{1}{2}} \pi^{-1}\left[W_{\lambda,\nu_1}(4k_1)\right]^{-1}$$

$$e^{2k_2} (4k_2)^{-\lambda}$$

$$x^{\nu_1+1} y^{\nu_2+1} y^{-(\nu_2+2)} \cdot \int_0^\infty e^{-k_1\left\{\frac{u}{x}+\left(\frac{x}{u}\right)^{\mu_1}\right\}} u^{-(\nu_1+2)} .$$

$$\{\alpha(u, y+)+\alpha(u, y-)\}\, du \cdot e^{-2k_2} (y\sqrt{\pi})\{2k_2\mu_2(\mu_2+1)\}^{-\frac{1}{2}} \text{ as } k_2 \to \infty$$

$$\sim \prod_1^2 \left[k_i^{\lambda+\frac{1}{2}} \{\mu_i(\mu_i+1)\}^{\frac{1}{2}} \cdot e^{2k_i} \cdot (4k_i)^{\lambda} \right] \cdot 2^{4\lambda-1} \cdot \pi^{-1} \cdot x^{\nu_1+1} \cdot y^{\nu_2+1}$$

$$x^{-(\nu_1+2)} y^{-(\nu_2+2)} e^{-2k_2} e^{-2k_1} [\alpha]_{x\pm,\, y\pm} \cdot (x\sqrt{\pi})(y\sqrt{\pi}) \cdot 2^{-1}$$

$$\prod_1^2 \left\{ k_i \mu_i (\mu_{i+1}) \right\}^{-\frac{1}{2}} ,$$

as $k_1 \to \infty$, $k_2 \to \infty$.

$$= \tfrac{1}{4}[\alpha]_{x\pm,\, y\pm}.$$

The same result holds when $k_2 \to \infty$, $k_1 \to \infty$.

This completes the theorem.

Remark : Similarity of the results of theorem 1.3.1. & 1.3.2 obviously reveals that such real inversion formulae may be generalised to functions of n variables.

The author expresses his grateful thanks to the referee for his valuable comments and suggestions.

REFERENCES

[1] Bernstein, D. L.—The double Laplace Integral—*Duke Math. J.*, 8 (1911).

[2] Erdelyi, A.—The inversion of Laplace transformation—*Math. Mag.* 29 (1950-51).

[3] Rooney, P. G.—A new representation and inversion theory for the Laplace transformation—*Canad. J. Math.*, Vol. (IV) (1952), 438-444.

[4] Widder, D. V.—The Laplace Transform—Princeton, 1946.

Received
4.5.81
Revised
31.12.81

Dept. of Pure Math.
Calcutta University