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FIXED POINT THEOREMS

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In this paper we shall prove two fixed point theorems which are extensions of a theorem of A. A. Ivanov [1] and a theorem of S. Reich [2].

Theorem 1. (Extension of Ivanov's theorem)

Let X be a non-empty metric space and $T : X \to X$, be a self-mapping of X. If X is T-orbitally complete, T is orbitally continuous and for every distinct x, y in X there exist real numbers $a_i (i=1, 2 \ldots 7)$ such that

$$(1) \quad a_1 d(x, y)+a_2 d(x, Tx)+a_3 d(y, Ty)+a_4 d(y, Tx)+a_5 d(x, Ty) + a_6 d(Tx, Ty)+a_7 \frac{d(x, Tx)\, d(y, Ty)}{d(x, y)} \geqslant 0,$$

where,

$$(2) \quad a_1+a_2+a_3+a_6+a_7 < \min \{0, -(a_4+a_5)\},$$

$$(3) \quad a_7+a_6+\frac{a_2+a_3}{2}+\frac{a_4+a_5}{2} < 0,$$

then T has a fixed point in X.

Proof : By the symmetric property of metric, we can easily obtain

$$(4) \quad a_1 d(x, y)+\frac{a_2+a_3}{2}\Big[d(x, Tx)+d(y, Ty) \Big]+\frac{a_4+a_5}{2}\Big[d(y, Tx)+d(x, Ty) \Big] + a_6 d(Tx, Ty)+a_7 \frac{d(y, Tx)\, d(y, T_y)}{d(x, y)} \geqslant 0.$$

Since x and y are arbitrary, let $y=Tx$. Then from (4) we have

$$(5) \quad a_1 d(x, Tx)+\frac{a_2+a_3}{2}\Big[d(x, Tx)+d(Tx, T^2x) \Big]+\frac{a_4+a_5}{2}\, d(x, T^2x) + a_6 d(Tx, T^2x)+a_7 d(Tx, T^2x) \geqslant 0.$$

Now we consider the following two cases :

Case (i) : When $a_4+a_5 \geqslant 0$, then $d(x, T^2x) \leqslant d(x, Tx)+d(Tx, T^2x)$ and we obtain by virtue of (5)

$$(6) \quad \left(a_1+\frac{a_2+a_3}{2}+\frac{a_4+a_5}{2}\right) d(x, Tx)+ \left(\frac{a_2+a_3}{2}+\frac{a_4+a_5}{2}+a_6+a_7\right) . d(Tx, T^2x) \geqslant 0.$$

Since $a_4+a_5 \geq 0$, it follows therefore from (2) and (3)

$$\frac{2a_1+a_2+a_3}{2}+\frac{a_4+a_5}{2}+\frac{a_2+a_3}{2}+\frac{a_4+a_5}{2}+a_6+a_7 < 0,$$

or, $\left(a_1+\frac{a_2+a_3}{2}+\frac{a_4+a_5}{2}\right)\left(\frac{a_2+a_3}{2}+\frac{a_4+a_5}{2}+a_6+a_7\right)^{-1}+1>0$

or, $-(2a_1+a_2+a_3+a_4+a_5)(a_2+a_3+a_4+a_5+2a_6+2a_7)^{-1} < 1.$

Now from (6), we have $(2a_1+a_2+a_3+a_4+a_5) \geq 0$

Thus we get

(7) $0 \leqslant -(2a_1+a_2+a_3+a_4+a_5)(a_2+a_3+a_4+a_5+2a_6+2a_7)^{-1} < 1.$

Combining (6) and (7) we have,

$$d(Tx, T^2x) \leqslant -(2a_1+a_2+a_3+a_4+a_5) .(a_2+a_3+a_4+a_5+2a_6+2a_7)^{-1} d(x, Tx).$$

By induction it may be shown that $\{T^n x\}_{n=0}^{\infty}$ is a Cauchy sequence. Since the metric space X is T-orbitally complete, $\lim_{n\to\infty} T^n x = u \in X$.

Next we shall show that u is a fixed point.

Since T is orbitally continuous, we have

$$Tu = T \lim_{n\to\infty} T^n x = \lim_{n\to\infty} T^{n+1} x = u,$$

which implies that u is a fixed point of T.

Case (ii) : When $a_4+a_5 < 0$, then

$d(x, T^2x) \geq d(Tx, T^2x) - d(x, Tx)$ and then we obtain by virtue of (5)

(8) $\left(a_1+\frac{a_2+a_3}{2}-\frac{a_4+a_5}{2}\right) d(x, Tx)+\left(\frac{a_2+a_3}{2}+\frac{a_4+a_5}{2}+a_6+a_7\right) .d(Tx, T^2x) \geq 0$

By similar argument of case (i), it may be easily shown that

$$-(2a_1+a_2+a_3-a_4-a_5)(a_2+a_3+a_4+a_5+2a_6+2a_7)^{-1} < 1$$

Thus from (8) we have

$$2a_1+a_2+a_3-a_4-a_5 \geq 0,$$

so that

$$0 \leqslant -(2a_1+a_2+a_3-a_4-a_5)(a_2+a_3+a_4+a_5+2a_6+2a_7)^{-1} < 1.$$

It follows therefore from (8) that

$$d(Tx, T^2x) \leq -(2a_1+a_2+a_3-a_4-a_5)(a_2+a_3+a_4+a_5+2a_6+2_7)^{-1}\, d(x, Tx).$$

The remaining part of the proof is similar to that of case (i).

Remark : Putting $a_7=0$ in theorem 1 we get the theorem (1) of Ivanov [1] as a particular case of our theorem.

Theorem 2. (Extension of Reich's theorem)

Let (X, d) be a complete metric space and $T : X \to X$ and let $t : X \to$ set of real numbers be defined by $t(x)=d(x, Tx)$. If for any $x, y \in X$,

(9) $d(Tx, Ty) < a_1\, t(x)+a_2\, t(y)+a_3\, d(x, y)+a_4\, d(x, Ty)+a_5\, d(y, Tx)$

where a_i's are non-negative real numbers and $a_3+a_4+a_5 < 1$,

(10) t is lower semi-continuous

(11) there exists a sequence $\{x_n\} \subset X$ such that

$t(x_n) \to 0$ as $n \to \infty$,

then T has a unique fixed point in X.

Proof : Let $\{x_n\}$ be any sequence with $t(x_n) \to 0$.

Now for $m > n$,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, Tx_n)+d(Tx_n, Tx_m)+d(x_m, Tx_m) \\ &\leq d(x_n, Tx_n)+d(x_m, Tx_m)+a_1\, t(x_n)+a_2\, t(x_m)+a_3\, d(x_n, x_m) \\ &\qquad +a_4\, d(x_n, Tx_m)+a_5\, d(x_m, Tx_n) \\ &\leq (1+a_1+a_5)\, t(x_n)+(1+a_2+a_4)\, t(x_m)+(a_3+a_4+a_5)\, d(x_n, x_m). \end{aligned}$$

Thus

$$d(x_n, x_m) \leq \frac{1+a_1+a_5}{1-a_3-a_4-a_5}\, t(x_n)+\frac{1+a_3+a_4}{1-a_3-a_4-a_5}\, t(x_m).$$

Since $t(x_n) \to 0$ as $n \to \infty$, there exists an integer N such that

$$t(x_n) \leq \frac{(1-a_3-a_4-a_5)\,\epsilon}{2(1+a_2+a_5)} \text{ and } t(x_m) \leq \frac{(1-a_3-a_4-a_5)}{2(1+a_3+a_4)}\,\epsilon,\quad n > N,$$

where $\epsilon > 0$. Thus $\{x_n\}$ is a Cauchy sequence.

Hence $x_n \to x \in X$. Since t is lower semi-continuous, so, $t(x)=0$ and then $Tx=x$. Uniqueness of the fixed point follows easily.

Remark : Putting $a_4 = a_5 = 0$, in Theorem 2 we get the theorem of S. Reich [2] as a special case of our theorem.

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