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ON A FIXED POINT THEOREM OF KIYOSHI

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1. Introduction : In a recent paper [6], I. Kiyoshi generalized a theorem of K. Goebel and E. Zlotkiewicz [3] in the following form :

Theorem 1. Let F be a mapping of a Banach space X into itself. If F satisfies the conditions

(1) $F^2=I$, where I is the identity mapping

(2) $\| F(x)-F(y) \| \leqslant \alpha \| x-y \| +\beta(\| x-F(x) \| + \| y-F(y) \|)$ for every $x, y \in X$, where $0 \leqslant \alpha, \beta$ and $0<\alpha+4\beta<2$,

then F has at least one fixed point.

It may be of interest to remark that the first part in the right member of (2) is involved in the contraction mapping of S. Banach [1] and the second part of (2) is contained in the contraction type mapping of R. Kannan [5]. Also S. K. Chatterjea [2] introduced another contraction type mapping by means of the metric relation

$$d(F(x), F(y)) \leqslant \alpha[d(x, F(y))+d(y, F(x))].$$

Finally G. E. Hardy and T. D. Rogers [4] introduced the concept of generalized contraction type mapping by means of the metric relation

$$d(F(x), F(y)) \leqslant \alpha_1 d(x, y)+\alpha_2 d(x, F(x))+\alpha_3 d(y, F(y))+\alpha_4 d(x, F(y))+\alpha_5 d(y, F(x)).$$

Noticing these various types of contraction type relations, we are led to consider an extension of the theorem of Kiyoshi in the following form :

Theorem 2. Let F be a mapping of a Banach space X into itself. If F satisfies the conditions

(i) $F^3=I$ where I is the identity mapping

(ii) $$\| F(x)-F(y) \| \leqslant \alpha \| x-y \| +\beta \| x-F(x) \| +\gamma \| y-F(y) \| + \delta \| x-F(y) \| +\epsilon \| y-F(x) \|$$

for every $x, y \in X$ where $0 \leqslant \alpha, \beta, \gamma, \delta, \epsilon$ and $\alpha^2+2\alpha(\beta+\gamma+\delta+\epsilon+1)+3(\beta+\gamma+\delta+\epsilon)<1$,

then F has at least one fixed point.

2. Proof of theorem 2. Let x be a point of X and we put $y=\frac{1}{2}(F+I)(x)$, $z=F(y)$, $u=2y-3$. Now by conditions (i) and (ii) we have

$$\| z-x \| = \| F(y)-F^3(x) \| \leqslant \alpha \| y-F^2(x) \| +\beta \| y-F(y) \| +\gamma \| F^2(x)-F^3(x) \| +\delta \| y-F^3(x) \| +\epsilon \| F^2(x)-F(y) \|$$

Again, by symmetry

$$\|z-x\| = \|F^3(x)-F(y)\| \leqslant \alpha \|F^2(x)-y\| + \beta \|F^2(x)-F^3(x)\| + \gamma \|y-F(y)\| + \delta \|F^2(x)-F(y)\| + \epsilon \|y-F^3(x)\|$$

Thus it follows from the above two relations that

$$\|z-x\| \leqslant \alpha \|y-F^2(x)\| + \frac{\beta+\gamma}{2} \|y-F(y)\| + \frac{\beta+\gamma}{2} \|F^2(x)-F^3(x)\| + \frac{\delta+\epsilon}{2} \|F^2(x)-F(y)\| + \frac{\delta+\epsilon}{2} \|y-F^3(x)\|$$

Now $\|y-F^2(x)\| \leqslant \|y-F(y)\| + \|F(y)-F^2(x)\|$

$$\leqslant \|y-F(y)\| + \alpha \|y-F(x)\| + \beta \|y-F(y)\| + \gamma \|F(x)-F^2(x)\| + \delta \|y-F^2(x)\| + \epsilon \|F(x)-F(y)\|$$

$$\leqslant \|y-F(y)\| + \frac{\alpha}{2} \|x-F(x)\| + \beta \|y-F(y)\| + \gamma \|F(x)-y\| + \gamma \|y-F^2(x)\| + \delta \|y-F^2(x)\| + \epsilon \|F(x)-y\| + \epsilon \|y-F(y)\|$$

$$\leqslant (1+\beta+\epsilon) \|y-F(y)\| + \tfrac{1}{2}(\alpha+\gamma+\epsilon) \|x-F(x)\| + (\gamma+\delta) \|y-F^2(x)\|$$

Also by symmetry,

$$\|y-F^2(x)\| = \|F^2(x)-y\| \leqslant \|F^2(x)-F(y)\| + \|F(y)-y\|$$

$$= \|y-F(y)\| + \|F^2(x)-F(y)\|$$

$$\leqslant \|y-F(y)\| + \alpha \|F(x)-y\| + \beta \|F(x)-F^2(x)\| + \gamma \|y-F(y)\| + \delta \|F(x)-F(y)\| + \epsilon \|y-F^2(x)\|$$

$$\leqslant \|y-F(y)\| + \frac{\alpha}{2} \|x-F(x)\| + \beta \|F(x)-y\| + \beta \|y-F^2(x) + \gamma \|y-F(y)\| + \delta \|F(x)-y\| + \delta \|y-F(y)\| + \epsilon \|y-F^2(x)\|$$

Thus we have,

$$\|y-F^2(x)\| \leqslant \frac{2+\beta+\gamma+\delta+\epsilon}{2-(\beta+\gamma+\delta+\epsilon)} \|y-F(y)\| + \frac{2\alpha+\beta+\gamma+\delta+\epsilon}{2\{2-(\beta+\gamma+\delta+\epsilon)\}} \|x-F(x)\|$$

and $\|F^3(x)-F^2(x)\| \leqslant \|y-F^3(x)\| + \|y-F^2(x)\|$

$$\therefore \quad \|z-x\| \leqslant \alpha \|y-F^2(x)\| + \frac{\beta+\gamma}{2} \|F^2(x)-y\| + \frac{\beta+\gamma}{2} \|y-F^3(x)\| + \frac{\beta+\gamma}{2} \|y-F(y)\| + \frac{\delta+\epsilon}{2} \|F^2(x)-y\| + \frac{\delta+\epsilon}{2} \|y-F(y)\| + \frac{\delta+\epsilon}{2} \|y-F^3(x)\|$$

Also, $\|u-x\| = \|F(x)-F(y)\| \leqslant \|F(x)-y\| + \|y-F(y)\|$

$$= \tfrac{1}{2} \|x-F(x)\| + \|y-F(y)\|$$

Now, $\| z-u \| \leqslant \| z-x \| + \| u-x \|$

and $\| z-u \| = 2 \| y-F(y) \|$

$$\therefore \quad 2 \| y-F(y) \| \leqslant \frac{2\alpha+\beta+\gamma+\delta+\epsilon}{2} \left[\frac{2+\beta+\gamma+\delta+\epsilon}{2-(\beta+\gamma+\delta+\epsilon)} \| y-F(y) \| \right.$$

$$\left. + \frac{2\alpha+\beta+\gamma+\delta+\epsilon}{2\{2-(\beta+\gamma+\delta+\epsilon)\}} \| x-F(x) \| \right] + \frac{\beta+\gamma+\delta+\epsilon}{2} \| y-F(y) \|$$

$$+ \frac{\beta+\gamma+\delta+\epsilon}{2} \| x-F(x) \| + \tfrac{1}{2} \| x-F(x) \| + \| y-F(y) \|$$

In other words, we have

$$\| y-F(y) \| \leqslant \frac{4\alpha^2+4\alpha(\beta+\gamma+\delta+\epsilon)+4}{2\{4-6(\beta+\gamma+\delta+\epsilon)-2\alpha(\beta+\gamma+\delta+\epsilon)-4\alpha\}} \| x-F(x) \|$$

Let $G=\frac{1}{2}(F+I)$. Now for any $x \in X$, we have

$$\| G^2(x)-G(x) \| = \| G(y)-y \| = \| \tfrac{1}{2}(F+I)(y)-y \| = \tfrac{1}{2} \| y-F(y) \|$$

$$\leqslant \frac{1}{2} \frac{4\alpha^2+4\alpha(\beta+\gamma+\delta+\epsilon)+4}{2\{4-6(\beta+\gamma+\delta+\epsilon)-2\alpha(\beta+\gamma+\delta+\epsilon)-4\alpha\}} \| x-F(x) \|$$

$$= \frac{1}{2} \frac{\alpha^2+\alpha(\beta+\gamma+\delta+\epsilon)+1}{2-3(\beta+\gamma+\delta+\epsilon)-\alpha(\beta+\gamma+\delta+\epsilon)-2\alpha} \| x-F(x) \|$$

$$= \frac{\alpha^2+\alpha(\beta+\gamma+\delta+\epsilon)+1}{2-3(\beta+\gamma+\delta+\epsilon)-\alpha(\beta+\gamma+\delta+\epsilon)-2\alpha} \| G(x)-x \| .$$

By the hypothesis, we have

$$0 \leqslant \left[\frac{\alpha^2+\alpha(\beta+\gamma+\delta+\epsilon)+1}{2-3(\beta+\gamma+\delta+\epsilon)-\alpha(\beta+\gamma+\delta+\epsilon)-2\alpha} \right] < 1.$$

Thus $\{G^n(x)\}$ is a Cauchy sequence in X. By the completeness $\{G^n(x)\}$ converges to some element x_0 in X, i.e. $\lim_{n\to\infty} G^n(x)=x_0$. Thus $G(x_0)=x_0$. Hence $F(x_0)=x_0$ is a fixed point of F. This completes the proof of our proposed theorem.

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