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ON A TYPE OF SEMI-SYMMETRIC CONNECTION ON A RIEMANNIAN MANIFOLD

M. C. CHAKI and ARABINDA KONAR

Introduction.

The present paper deals with a type of semi-symmetric metric connection ∇ on a Riemannian Manifold such that the curvature tensor R and the torsion tensor T of ∇ satisfy the conditions 1) $R(X, Y)Z=0$ and 2) $(\nabla_X T)(Y, Z) = B(X)\, T(Y, Z)$, where B is a 1-form. The nature of curvature restriction on the manifold induced by the introduction of such a connection is determined. Further, it is shown that if for a semi-symmetric metric connection the manifold is a group manifold, then the manifold is of constant curvature.

1. **Preliminaries.** Let (M, g) be an n-dimensional Riemannian manifold with Levi-civita connection $\overset{o}{\nabla}$. A linear connection on (M, g) is said to be semi-symmetric if

$$T(X, Y)=\pi(Y)X-\pi(X)Y \quad [1] \qquad \dots (1.1)$$

where π is a 1-form.

Then we have

$$\nabla_X Y=\overset{o}{\nabla}_X Y+\pi(Y)X-g(X, Y)\rho \qquad \dots (1.2)$$

where $g(X, \rho)=\pi(X)$ for every vector field X. .. (1.3)

Further, if R and K denote the curvature tensors of ∇ and $\overset{o}{\nabla}$ respectively, then

$$R(X, Y)Z=K(X, Y)Z-\alpha(Y, Z)X+\alpha(X, Z)Y -g(Y, Z)AX+g(X, Z)AY \qquad \dots (1.4)$$

where α is a tensor field of type (0, 2) defined by

$$\alpha(X, Y)=(\overset{o}{\nabla}_X\pi)(Y)-\pi(X)\pi(Y)+\tfrac{1}{2}\pi(\rho)g(X, Y) \qquad \dots (1.5)$$

and A is a tensor field of type (1, 1) defined by

$$g(AX, Y)=\alpha(X, Y) \qquad \dots (1.6)$$

for any vector fields X, Y.

Moreover, we have

$$(\nabla_X\pi)(Y)=(\overset{o}{\nabla}_X\pi)(Y)-\pi(X)\pi(Y)+\pi(\rho)\, g(X, Y) \qquad \dots (1.7)$$

We shall use the above results in the next section.

2. A Special type of semi-symmetric connection

We consider a type of semi-symmetric metric connection ∇ whose curvature tensor R and torsion tensor T satisfy the following conditions :

$$R(X, Y)Z=0 \quad \ldots (2.1)$$

and

$$(\nabla_X T)(Y, Z)=B(X)\, T(Y, Z) \quad \ldots (2.2)$$

where B is a 1-form.

From (1.1) we have

$$(C_1^1 T)(Y)=(n-1)\, \pi(Y) \quad \ldots (2.3)$$

From (2.3) we get

$$(\nabla_X C_1^1 T)(Y)=(n-1)(\nabla_X \pi)(Y) \quad \ldots (2.4)$$

Again from (2.2) we obtain

$$(\nabla_X C_1^1 T)(Y)=B(X)(C_1^1 T)(Y)=B(X)(n-1)\, \pi(Y)$$

$$=(n-1)\, B(X)\, \pi(Y) \quad \ldots (2.5)$$

From (2.4) and (2.5) we get

$$(\nabla_X \pi)(Y)=B(X)\, \pi(Y) \quad \ldots (2.6)$$

Using (2.6) we can express (1.7) as follows :

$$B(X)\, \pi(Y)=(\overset{0}{\nabla}_X \pi)(Y)-\pi(X)\pi(Y)+\pi(\rho)\, g(X, Y)$$

Hence

$$(\overset{0}{\nabla}_X \pi)(Y)=B(X)\, \pi(Y)+\pi(X)\, \pi(Y)-\pi(\rho)\, g(X, Y) \quad \ldots (2.7)$$

Using (2.7) it follows from (1.5) that

$$\alpha(X, Y)=B(X)\, \pi(Y)-\tfrac{1}{2}\pi(\rho)\, g(X, Y) \quad \ldots (2.8)$$

Now,

$$g(AX, Y)=\alpha(X, Y)=B(X)\, \pi(Y)-\tfrac{1}{2}\pi(\rho)\, g(X, Y) \quad \text{by } (2.8)$$

$$=B(X)\, g(\rho, Y)+g(-\tfrac{1}{2}\pi(\rho)X, Y)=g(B(X)\rho, Y)$$

$$+g(-\tfrac{1}{2}\pi(\rho)X, Y)$$

$$=g(B(X)\rho-\tfrac{1}{2}\pi(\rho)X, Y)$$

Hence

$$AX=B(X)\, \rho-\tfrac{1}{2}\pi(\rho)\, X \quad \ldots (2.9)$$

Using (2.8) and (2.9) we can write (1.4) as follows :

$$R(X, Y)Z = K(X, Y)Z + B(X)\,[\pi(Z)Y - g(Y, Z)\rho] + B(Y)\,[-\pi(Z)X + g(X, Z)\,\rho] + \pi(\rho)\,[g(Y, Z)X - g(X, Z)\,Y] \quad \ldots (2.10)$$

Since by (2.1), $R(X,Y)Z=0$ it follows from (2.10) that

$$K(X, Y)Z = B(X)\,[g(Y, Z)\rho - \pi(Z)\,Y] - B(Y)\,[g(X, Z)\rho - \pi(Z)\,X] - \pi(\rho)\,[g(Y, Z)X - g(X, Z)\,Y] \quad \ldots (2.11)$$

Hence we can state the following theorem :

Theorem 1. If a Riemannian manifold admits a semi-symmetric metric connection whose curvature tensor R and torsion tensor T satisfy the conditions (2.1) and (2.2), then the curvature tensor of the manifold is given by (2.11).

Let us now assume that $B=0$. Then (2.2) takes the form

$$(\nabla_X T)\,(Y, Z) = 0 \quad \ldots (2.12)$$

Thus when (2.1) and (2. 12) are satisfied, (2.11) assumes the form

$$K(X, Y)Z = -\pi(\rho)\,[g(Y, Z)X - g(X, Z)\,Y] \quad \ldots (2.13)$$

From (2.13) it follows that the manifold is of constant curvature.

Hence we have the following Theorem :

Theorem 2. If a Riemannian manifold admits a semi-symmetric metric connection for which the conditions (2.1) and (2.12) are satisfied, then the manifold is of constant curvature.

Now in virtue of (2.1) and (2.12) it follows from a known result [3] that the connection ∇ determines a simply transitive group. Since in such a case the manifold is called a group manifold, the above theorem may be stated alternatively as follows :

If a Riemannian metric admits a semi-symmetric metric connection for which the manifold is a group manifold, then the manifold is of constant curvature.

Theorem 2 expressed in its alternative form was proved by Yano in [3].

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Dept. Pure Math.
Calcutta University
and
Dept. Math.,
Presidency College,
Calcutta