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# ON THE CONVERGENCE OF LAPLACE INTEGRAL

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## 1. Introduction.

Consider $Q$: the $n$ dimensional Euclidean space $R_n$ of elements $\mathbf{x} = (x_1, \ldots, x_n)$, $x_k \geqslant 0$. Then $\Gamma = C \cup \{\phi\}$, $C$ set of cells $C: \{\mathbf{x} : a_i < x_i < b_i$, $i = 1, 2, \ldots, n\}$ and $\{\phi\}$, the null set, is a semi-ring. If for a cell $U = (a_1, b_1; \ldots; a_n, b_n)$, $\mu(\phi) = 0$ and $\mu(U) = \prod_{j=1}^{n} (b_j - a_j)$, then $\mu$ is a measure on $\Gamma$. By means of the extension procedure we then obtain the $\sigma$-ring $\Delta$ of all $\mu$-measurable sets $A$, where $A$ is a Lebesgue measurable kernel on $Q$. The $n$-dimensional Lebesgue measure $\mu = \mu(A)$ (countably additive), the measure of $A$, is totally $\sigma$-finite and $(Q, A, \mu)$ constitutes a totally $\sigma$-finite measure space.

Let $F$ and $F_0$ be two $A$-measurable functions

$$\{x \in A : a < F,\ F_0 \leqslant \infty,\ a, \text{ any real number} > 0\}$$

defined on $Q$ such that

$$F_0 = F \text{ on } A \quad \text{and} \quad F_0 = 0 \text{ on } Q \setminus A.$$

Let $F_0$ be Lebesgue summable on $\Omega : a \leqslant x_j \leqslant X_j$, $j = 1, 2, \ldots, n$, a subspace of $A$ and $\mathbf{p} = (p_1, \ldots, p_n)$, where $p_j = \sigma_j + i\lambda_j$, $i = (-1)^{\frac{1}{2}}$, $j = 1, 2, \ldots, n$, are independent complex parameters. Let $\mathbf{p}.\mathbf{x}$ denote the inner product of $\mathbf{p}$ and $\mathbf{x}$ and

$$(1.1) \quad f(X; \mathbf{p}) = \int_{\Omega} e^{-\mathbf{p}.\mathbf{x}} F d\mu, \quad \mathbf{x} \in \Omega \subset Q.$$

If $\lim_{X \to \infty} f(X; \mathbf{p})$ exists, we say that the $n$-dimensional Laplace integral

$$(1.2) \quad \int_0^{\infty} e^{-\mathbf{p}.\mathbf{x}} F d\mu$$

converges at $\mathbf{p}$ to the value

$$(1.3) \quad f(\mathbf{p}) = \lim_{X \to \infty} f(X; \mathbf{p}).$$

The Laplace integral (1.2) is said to converge boundedly at $\mathbf{p}$, if (1.2) exists and

$$(1.4) \quad | f(X; \mathbf{p}) | \leqslant M,$$

where $X \in A$ and $M$ is independent of $X$. (1.1) may be termed a section of the Laplace integral (1.2).

It follows from the example given by Ditkin and Prudnikov ([4], P.7) as also from examples by Voelker and Doetsch [6] and Bernstein and Coon [2] that the convergence of a Laplace integral on a Euclidean plane at a point $(p, q)$ does not imply convergence of all its sections at the same point $(p, q)$. The same result obviously holds for the $n$-dimensional Laplace integral and the mere convergence of (1.2) at a point $\mathbf{p}_0$ does not imply convergence of (1.1) for all $X$ at the same point $\mathbf{p}_0$. Hence for the existence of the $n$-dimensional Laplace integral we must ensure the simultaneous existence of (1.1) and (1.2) at the same point $\mathbf{p}$. As such we are lead to consider the bounded convergence of (1.2).

## 2. Bounded convergence of (1.2).

Let $\phi(\mathbf{x})$ represent the indefinite integral

$$(2.1) \qquad \phi(\mathbf{x}) = \int_E e^{-\mathbf{p}_0.\mathbf{x}} F d\mu$$

where $E \subset \Omega$ and $\mathbf{p}_0 = (p_{10}, \ldots, p_{n0})$, $p_{j0} = \sigma_{j0} + i\lambda_{j0}$.

Since $e^{-\mathbf{p}_0.\mathbf{x}} F$ is summable on $\Omega$ and therefore on $E$, it follows that $\phi(x)$ is finite and is absolutely continuous with respect to

$$\mu: \ \phi \ll \mu \quad \text{and} \quad d\phi = e^{-\mathbf{p}_0.\mathbf{x}} F d\mu.$$

Let $h = (h_1, h_2, \ldots, h_n) = \mathbf{p} - \mathbf{p}_0$, so that $h_j = p_j - p_{j0} = \sigma_j - \sigma_{j0} + i(\lambda_j - \lambda_{j0}) = a_j + ib_j$, say. Thus $a_j > 0$, when re $\mathbf{h} = \text{re}(\mathbf{p} - \mathbf{p}_0) > 0$.

Then (1.1) can be written in the form

$$(2.2) \qquad f(X \,;\, \mathbf{p}) = \int_\Omega e^{-\mathbf{h}.\mathbf{x}} d\phi$$

where re $\mathbf{h} > 0$.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_{i,j,\ldots,l} = (\alpha_i, \alpha_j, \ldots, \alpha_l)$ with similar notations for $\beta$.

$\Delta F$ represents the $n$-dimensional increment of $F(\mathbf{x})$ from $F(\alpha)$ to $F(\beta)$, which means a summation of which the first term is $F(\beta)$ and the subsequent terms are obtained from this by putting one or more of the $\beta_j$ equal to $\alpha_j$ and then effecting a change of sign for each such substitution.

By $\Delta_{i,j,\ldots,l}(F)$ we mean $\Delta F$ in which the $i$ th, the $j$ th, ..., the $l$ th terms have been omitted from $\alpha$ and $\beta$.

Then Young's formula for integration by parts [7] reads as follows :

$$(2.3)\quad \int_{\alpha}^{\beta} g.\,df = \Delta(f.g) - \sum_{i} \int_{\alpha_i}^{\beta_i} \Delta_i(f.dg) + \sum_{i,j} \int_{\alpha_{i,j}}^{\beta_{i,j}} \Delta_{i,j}\,(f.dg)$$

$$- \sum_{i,j,k} + \ldots + (-1)^n \int_{\alpha}^{\beta} g.df$$

We apply (2.3) to (2.2) and make use of the property that $\phi$ vanishes when $x_j = 0,\ 0 \leqslant x_j \leqslant X_j$. We then obtain

$$f(X\,;\,p) = \exp(-h.X)\,\phi(X) + \sum_{i} h_i \exp(-h.X + h_i X_i) \int_0^{x_i} \phi(X\,;\,x_i) \exp(-h_i x_i)\,dx_i$$

$$+ \sum_{i,j} h_i h_j \exp(-h.X + h_{i,j} X_{i,j}) \int_0^{X_{i,j}} \phi(X\,;\,x_i, x_j) \exp(-h_{i,j} x_{i,j})\,dx_{i,j}$$

$$(2.4)\quad + \sum_{i,j,k} + \ldots + H \int_0^{X} \phi(X) \exp(-h.x)\,dx$$

where $H = h_1 \ldots h_n$, $dx = dx_1 \ldots dx_n$, $dx_{i,j,\ldots,l} = dx_i\, dx_j \ldots dx_l$, and $\phi(X_j, x_i, x_j, \ldots, x_l)$ is the expression $\phi(X)$ in which the elements $X_i, X_j, \ldots, X_l$ of $X$ are replaced by $x_i, x_j, \ldots, x_l$.

If the sections (1.1) of the Laplace integral (1.2) satisfy (1.4) at $p = p_o$, then $\phi$ is uniformly bounded : $|\phi| \leqslant M$, independent of the $x_j$. Hence from (2.4) after some easy steps

$$(2.5)\quad |f(X\,;\,p)| \leqslant M\left(1 + \sum_{i} \frac{|h_i|}{a_i} + \sum_{i,j} \frac{|h_i||h_j|}{a_i a_j} + \ldots + \frac{|h_1| \ldots |h_n|}{a_1 \ldots a_n}\right)$$

It follows, therefore, from (2.5) that if the Laplace integral (1.2) converges boundedly at $p_o$, the Laplace integral converges boundedly in the region for which re $(p - p_o) \geqslant 0$.

It may be noted that for the validity of the above result it is necessary that both the conditions (1.3) and (1.4) must have to be satisfied simultaneously at $p_o$. (See the example on P.7, Ditkin and Prudnikov [4] for the corresponding result in connection with the two dimensional Laplace integral.)

**3. Absolute Convergence of (1.2).**

The Laplace integral (1.2) is absolutely convergent, if $\int_0^{\infty} |e^{-p.x} F|\, d\mu$ exists finitely.

$$\text{Since } \left| \int_{\Omega} e^{-\mathbf{p}.\mathbf{x}} F d\mu \right| \leqslant \int_{\Omega} \left| e^{-\mathbf{p}.\mathbf{x}} F \right| d\mu \leqslant \int_{0}^{\infty} \left| e^{-\mathbf{p}.\mathbf{x}} F \right| d\mu,$$

it follows that the absolute convergence of the Laplace integral (1.2) at $\mathbf{p}_o$ implies the bounded convergence of the integral at the same point.

Let $\mathbf{p} \equiv (p_1, .., p_n)$ and $\mathbf{p}_o \equiv (p_{1o}, ..., p_{no})$ be real and let the Laplace integral (1.2) be absolutely convergent when $\mathbf{p} = \mathbf{p}_o$ (fixed). Then, evidently, the Laplace integral is absolutely convergent for real $p_i$, $p_j$, where $i \neq j = 1, 2, ..., n$ for fixed values $p_i = p_{io}$, $p_j = p_{jo}$. If $p_i, p_j$ are complex and $\alpha_i(p_{jo})$ and $\alpha_j(p_{io})$ are the convergence abscissae (see Voelker and Doetsch [6] or the characteristics of convergence—See Ditkin and Prudnikov [4], p.9) with respect to $p_i$, $p_j$, respectively, and if $C_i$ and $C_j$ represent the domains

$$C_i : \text{ re } p_i \geqslant p_{io} > A_i \text{ ; re } p_j > \alpha_j (p_{io})$$

$$\text{and} \quad C_j : \text{ re } p_i > \alpha_i(p_{jo}) \text{ ; re } p_j \geqslant p_{jo} > A_j$$

($A_i, A_j$ are fixed real numbers).

Then as in Voelker and Doetsch [6], the domain of absolute convergence of the Laplace integral (1.2) with respect to $p_i$, $p_j$ is determined by

$$D_{i,j} = C_i \cup C_j$$

We now give to $i, j$ all possible values $i \neq j = 1, 2, ..., n$. Then when the Laplace integral (1.2) is absolutely convergent at the real point $\mathbf{p}_o$, the domain determined by the complex values $\mathbf{p}$ for which (1.2) is absolutely convergent, is the set

$$\{ \cup D_{i,j} : \ i \neq j = 1, 2, ..., n \}.$$

Let $D(A)$ represent the set of points $\mathbf{p}$ for which re $\mathbf{p} \geqslant A$, $A = (A_1, ..., A_n)$, with $D[A]$ as the closure. Then $D[A]$ is uniquely determined, when one assumes that the curves $p_i = \alpha_i(p_j)$ and $p_j = \alpha_j(p_i)$ in the real $p_i$-$p_j$ plane are, respectively, parallel to the $p_j$ and $p_i$ axes. For simplicity of our discussion we can choose $D[A]$ as the domain of absolute convergence of the Laplace integral (1.2). $D[A]$ then also represents the domain of bounded convergence of (1.2).

## 4. Uniform Convergence of (1.2).

Let $L(X,Y)$ be the line segment

$$L(X,Y) = \{ z \mid z = \theta X + (1-\theta) Y, 0 < \theta < 1 \}$$

joining two points $X$, $Y$ of $A$.

Put $D_r \psi(x ; z_r) = \dfrac{\partial}{\partial z_r} \psi (x ; z_r)$

and grad $\psi(x ; z) = (D_1 \psi(x ; z_1), ..., D_n \psi (x ; z_n))$.

Then if $\psi(X)$ have a differential everywhere in a neighbourhood $N(X)$ of $X \in A$ and $Y$ be a point which $\in N'(x)$, we have the mean value theorem

(4.1) $\psi(Y) - \psi(X) = (Y - X).\ \text{grad}\ \psi(X\,;\,z)$

(See Apostol [1] ; p. 135)

where $z \equiv (z_1, \ldots, z_n)$, $z_r \in L(x_r, y_r)$, $r=1,2,\ldots,n$.

If all the $D_r\psi(x\,;\,z_r)$ be uniformly bounded, such that

$\max|D_r\psi| = \frac{M_1}{n}$, $r=1,\ldots,n$ and $\max|Y-X| = \delta$, we have

(4.2) $$|\psi(Y)-\psi(X)| \leqslant M_1\delta.$$

Let $E_{rs}$ be the domain

(4.3) $$E_{rs}: \begin{cases} 0 \leqslant x_s \leqslant X_s, & \text{for } 1 \leqslant s \leqslant r-1 \\ Y_r \leqslant x_r \leqslant X_r, & \text{for } s=r \\ 0 \leqslant x_s \leqslant Y_s, & \text{for } r < s \leqslant n \end{cases}$$

Also let $\Omega_X : 0 \leqslant x_i \leqslant X_i$, $\Omega_Y : 0 \leqslant y_i \leqslant Y_i$, $i=1,2,\ldots,n$, be two hyper-rectangles having one common corner at the origin and the common adjacent sides through the origin along the co-ordinate axes, the end point of the diagonal through the origin of one being at $X=(X_1,\ldots,X_n)$, while that of the other at $Y=(Y_1,\ldots,Y_n)$. Then by considering the space included within $\Omega_X$, $\Omega_Y$, we obtain, if $X>Y$, the following decomposition

$$\Omega_X - \Omega_Y = \sum_{r=1}^{n} E_{rs}.$$

If now $\psi$ is integrable on $\Omega_X$, $\psi$ is integrable on $\Omega_Y$ and also on each $E_{rs}$ and we have

(4.4) $$\left(\int_{\Omega_X} - \int_{\Omega_Y}\right)\psi(x)d\mu = \sum_{r=1}^{n}\int_{E_{rs}}\psi d\mu$$

( See Kolmogorov and Fomin [5], P. 298)

Since $\phi \leqslant \mu$, exp. $(-\mathbf{h}x.)\,\phi(x)$ is also so and

(4.5) $|D_r(\exp(-\mathbf{h}.x)\phi(x)| \leqslant M_2$,

$r=1,2,\ldots,n$ in $\Omega: 0 \leqslant x_j \leqslant X_j$, $j=1,2,\ldots,n$.

Also when the sections of the Laplace integral (1.2) are bounded at $p_0$,

(4.6) $$|\phi(x)| \leqslant M_3$$

for $x_j \geqslant 0$, $j=1,2,\ldots,n$.

We now apply the formula (2.4) to the functions $f(X\,;\,\mathbf{p})$ and $f(Y\,;\,\mathbf{p})$, $X>Y$, simplify each term in $f(X\,;\,\mathbf{p}) - f(Y\,;\,\mathbf{p})$ by the mean value theorem (4.1), make use of the formula of type (4.4) holding when $n=2,3,\ldots$ and utilize relations of type (4.2), (4.5) and (4.6) as and when necessary. Then after some easy reductions, it follows, when $Y$ is large enough, that

$$| f(X;\mathrm{p}) - f(Y;\mathrm{p}) |$$

$$\leqslant M\Delta\Big[e^{-aY} + \sum_i \frac{|h_i|}{a_i} e^{-a_iY_i} + \sum_{i,j} \frac{|h_i|\,|h_j|}{a_i a_j}\left(e^{-a_iY_i}+e^{-a_jY_j}\right)$$

$$(4\cdot7) \qquad + \ldots + \frac{|H|}{\alpha}\left(\sum_{i=1}^{n} e^{-a_iY_i}\right)\Big],$$

where $\Delta = \max_j | X_j - Y_j |$, $M$ = max. of all the constants involved in the process, $\alpha = a_1 \ldots a_n$ and $H = h_1 \ldots h_n$.

We can assume that the hyper-rectangles constructed above with corners at $X$, $Y$ and common corner at $O,(X_i > Y_i)$ are such that $X$ is a linear function of $Y$. Then although $\Delta$ may tend to infinity with $Y_j$, $\Delta\exp(-a_jY_j)$ must tend to zero with $Y_j$.

Let $| h_i | \leqslant K_i a_i$, where $K_i$ are finite constants and $a_i \geqslant \delta_i > 0$. Then (4.7) reduces to

$$(4.8) \quad | f(X;p) - f(Y;p) | \leqslant M\Delta\Big[e^{-\delta_iY_i} + \sum_i K_i e^{-\delta_iY_i} + \ldots + \mathrm{k}\sum_{i=1}^{n} e^{-\delta_iY_i}\Big],$$

$\mathrm{k} = k_1 \ldots k_n$.

The right hand side of (4.8) is independent of $\mathbf{p}$ and can be made less than $\epsilon$ by taking $Y$ large enough.

The Laplace integral is therefore uniformly convergent for

$| h_i | \leqslant k_i a_i$, $a_i \geqslant \delta_i > 0$ i.e. for $| p_i - p_{i0} | \leqslant k_i(\sigma_i - \sigma_{i0})$;

$\sigma_i \geqslant \sigma_{i0} + \delta_i$, $i = 1,2,\ldots,n$, $k_i$ are positive constants which may be as large as we please.

We thus have the following theorem.

**Theorem :** Given that the Laplace integral (1.2) is absolutely convergent at $p_i = p_{i0}$. Then the Laplace integral is (absolutely and also) boundedly convergent for re $p_i \geqslant$ re $p_{i0}$ and is uniformly convergent for re $p_i \geqslant$ re $p_{i0} + \delta_i$, $| p_i - p_{i0} | \leqslant k_i$ re $(p_i - p_{i0})$, where $i = 1, 2, \ldots, n$ and the $k_i$ are real constants which may be as large as possible.

Thus the Laplace integral (1.2) which is absolutely convergent in $D[A]$, is boundedly convergent in $D[A]$ and in any compact region $\subset D[A]$, not containing $A$, the integral is uniformly convergent.

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