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L^2-CLASSIFICATION OF A VECTOR-MATRIX DIFFERENTIAL EQUATION

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1. Introduction

Let M denote the formally symmetric, second-order matrix differential expression given by, for suitably differentiable real-valued vector function $f=(f_1, f_2)^T$,

$$M[f]=\begin{bmatrix} -\frac{d}{dx}\left(p\frac{d}{dx}\right)+q_1 & q_2 \\ q_2 & -\frac{d}{dx}\left(r\frac{d}{dx}\right)+q_3 \end{bmatrix} f \qquad \text{on } [a, b) \qquad (1.1)$$

where the coefficients p, r and q_j $(j=1, 2, 3)$ satisfy the following conditions

(i) $p(x)$ and $r(x)$ are real-valued and absolutely continuous on all compact sub-intervals of $[a, b)$ and $p(x), r(x)>0$ $\quad (x \in [a, b))$

(ii) q_j $(j=1, 2, 3)$ are real-valued and continuous on $[a, b)$ with $q_1>0$ and $q_1q_3-q_2^2 \geqslant 0$

$$-\infty < a < b \leqq \infty .$$

Moreover if $\frac{1}{pr}$, q_1, q_2, q_3 are summable in the whole interval $[a, b)$ then the differential expression $M[f]$ is said to be regular at all points of $[a, b)$ i.e if $\xi \in [a, b)$ then the intial value problem

$$M[f]=0$$
$$f_1(\xi)=A, \quad (pf'_1)'(\xi)=C$$
$$f_2(\xi)=B, \quad (rf'_2)'(\xi)=D$$

on $[a, b)$ can be solved for arbitrary constants A, B, C, D: For this result see the existence Theorem 3.1, Sen Gupta [4]; otherwise, $M[\cdot]$ is said to be singular at the open end-point b(or if $b=\infty$).

The vector function $U=(u, v)^T$ is said to be a solution of (1.1) if u, v, pu' and rv' are absolutely continuous on all compact sub-intervals of $[a, b)$ and

$$\left.\begin{aligned} -(pu')'+q_1u+q_2v=0 \\ -(rv')'+q_2u+q_3v=0 \end{aligned}\right\}$$

2. Preliminaries

The Green's formula, for any two vector functions $f=(f_1, f_2)^T$ and $g=(g_1, g_2)^T$ sufficiently smooth, takes the form

$$\int_a^b \{f^T M[g] - g^T M[f]\} dx = [fg](b) - [fg](a)$$

when the bilinear form $[fg]$ (.) is given by

$$[fg](x) = p(x) f_1(x) g'_1(x) - p(x) f'_1(x) g_1(x) + r(x) f_2(x) g'_2(x) - r(x) f'_2(x) g_2(x).$$

It is well known that, if f, g are the solutions of (1.1) then $[fg]$ ($\cdot$) is independent of x.

3. L^2-classification

A vector function $f(x)$ which satisfies the differential system (1.1) is said to be a L^2-solution of (1.1) if

$$\int_0^\infty f^T f \, dx < \infty$$

holds. (i.e when each element of the vector function is square-integrable)

It was proved in Chakravarty [1, 2] and Sen Gupta [4, 5] that the differential system of the type (1.1) i.e a pair of second order differential systems can have at least 2 and atmost 4 L^2-solutions.

$M[\cdot]$ is said to be in the limit -2, 3 or 4 at infinity according as (1.1) has 2, 3 or 4 linearly independent solutions in $L^2(0, \infty)$, (the Hilbert space of vector functions with integrable square).

Theorem I. Let $N(x)$ be a positive, non-decreasing function such that

(i) $\int^\infty \frac{dx}{\sqrt{(pN)}}, \int^\infty \frac{dx}{\sqrt{(rN)}}$ diverges (3.1)

(ii) $\overline{\lim}_{x\to\infty} \frac{N' \sqrt{p}}{\sqrt{(N^3)}}$ converges, (3.2)

further, for all sufficiently large values of x

let $\frac{q_1 q_3 - q_2^2}{q_1} > -K N(x)$ (3.3)

(K is a positive constant)

Then the differential system (1.1) *is not limit*—4.

Proof. To prove the theorem it is sufficient to show that the differential system

$$M[U]=0 \tag{3.4}$$

has at least one solution not belonging to $L^2(0, \infty)$.

Multiplying the equation (3.4) by $U^T=(u, v)$ and dividing by N we get

$$-\frac{q_1u^2+2q_2uv+q_3v^2}{N}=-\frac{(pu')'u+(rv')'v}{N}$$

Integrating both sides,

$$-\int_a^x \frac{q_1u^2+2q_2uv+q_3v^2}{N}\,dt=-\left[\frac{puu'+rvv'}{N}\right]_a^x+\int_a^x \frac{pu'^2+rv'^2}{N}\,dt-\int_a^x \frac{(puu'+rvv')N'}{N^2}\,dt \tag{3.5}$$

But,

$$-\int_a^x \frac{q_1u^2+2q_2uv+q_3v^2}{N}\,dt = -\int_a^x \frac{1}{N}\left\{q_1\left(u+\frac{q_2}{q_1}v\right)^2+\frac{q_1q_3-q_2^2}{q_1}v^2\right\}dt$$

$$< K\int_a^x v^2dt < K\int_a^\infty v^2dt \qquad \text{[using (3.3)]}$$

$$=K_1 \text{ (say)} \qquad \text{[Supposing } U\in L^2[O, \infty)\text{]}.$$

Hence from (3.5)

$$K_1 > -\left[\frac{puu' + rvv'}{N}\right]_a^x+\int_a^x\frac{pu'^2+rv'^2}{N}\,dt-\int_a^x\frac{(puu'+rvv')N'}{N^2}dt, \ (\forall x) \tag{3.6}$$

We now show that if the solution $(u, v) \in L^2\ [O, \infty)$,

then the integral

$$\int_a^\infty \frac{pu'^2+rv'^2}{N}\,dt \text{ converges.}$$

Conversely, suppose that this integral diverges. Then the function

$$H(x)=\int_a^x \frac{pu'^2+rv'^2}{N}\,dt \tag{3.7}$$

is positive, monotonically increasing and tends to $+\infty$ as $x \to \infty$.

Now using Cauchy—Buniakovski inequality and the condition (3.2), we have, for a sufficiently large 'a' and for $x > a$

$$\left| \int_a^x \frac{puu'+rvv'}{N^2} N'\,dt \right| < \int_a^x \left\{ \left(\sqrt{\frac{p}{N^3}}\, N'\, u\right)^2 + \left(\sqrt{\frac{r}{N^3}}\, N' v\right)^2 \right\}^{1/2} \left\{ \frac{pu'^2+rv'^2}{N} \right\}^{1/2} dt$$

$$< K_2 \int_a^x (u^2+v^2)^{1/2} \left\{ \frac{pu'^2+rv'^2}{N} \right\}^{1/2} dt$$

$$< K_2 \left\{ \int_a^x (u^2+v^2)\,dt \right\}^{1/2} \left\{ \int_a^x \frac{pu'^2+rv'^2}{N}\,dt \right\}^{1/2}$$

$$< K_3 \sqrt{H(x)}$$

where K_2 is a certain constant depending on p, r and N, and

$$K_3 = K_2 \left(\int_0^\infty (u^2+v^2)\,dt \right)^{1/2}$$

Applying these results in (3.6) we get

$$K_1 > H(x) - \left[\frac{puu'+rvv'}{N} \right]_a^x - K_3 \sqrt{H(x)} \qquad (\forall\, x),$$

Since $H(x) \to \infty$ as $x \to \infty$, the last inequality can hold only if

$$\frac{puu'+rvv'}{N} > 0 \qquad \text{for large } x$$

i.e. $\quad puu'+rvv' > 0 \qquad$ (as $N > 0$)

or, $\quad \frac{p}{r} uu' > -vv' \qquad (r > 0)$

Two cases may arise :

Case (i) u and u' are of opposite sign.

Then $\frac{p}{r}uu'$ is negative since $\frac{p}{r} > 0$, and hence vv' is positive, which indicates that v and v' have the same sign for sufficiently large x.

Case (ii) u and u' have the same sign.

Thus either u and u' or v and v' are of the same sign. In either case one of the two integrals

$$\int^\infty u^2\, dx \text{ and } \int^\infty v^2\, dx$$

fails to exist, contradictory to the hypothesis $U \in L^2\,[0, \infty)$:

Thus

$$\int_0^\infty \frac{pu'^2+rv'^2}{N}\,dx \quad \text{exist for } U=(u,v)^T \in L^2\,[0,\infty)$$

so that

$$\sqrt{\frac{p}{N}}\,u',\ \sqrt{\frac{r}{N}}\,v' \in L^2\,[0,\infty). \tag{3.8}$$

Now let $F_j(x,\lambda)=(f_j(x,\lambda),\,g_j(x,\lambda))^T$, $j=1,2,3,4$, be the four linearly independent square-integrable solutions of the system $M[f]=\lambda f$. It is well known that $P_{jk}=[f_j(x,\lambda)\,f_k(x,\lambda)]$, $j,k=1,2,3,4$; $j\neq k$ is an integral function of λ independent of x. The Wronskian for the system is then given by

$$W(\lambda) \equiv W(F_1,F_2,F_3,F_4) = P_{12}\,P_{34}-P_{13}\,P_{24}+P_{14}\,P_{23}$$

which is equal to some constant c (not equal to zero), since the four solutions F_1, F_2, F_3, F_4 are linearly independent. Therefore at least one of the $P_{jk}\neq 0$. Say $P_{12}=k\neq 0$

i.e.
$$p\,f_1\,f_2'-p\,f_1'\,f_2+r\,g_1\,g_2'-r\,g_1'\,g_2 = k \tag{3.9}$$

case (i) if $p>r$, dividing both sides of (3.9) by $\sqrt{pN}$ and taking moduli we obtain

$$\sqrt{\frac{p}{N}}\,|f_1|\,|f_2'|+\sqrt{\frac{p}{N}}\,|f_1'|\,|f_2|+\frac{r}{\sqrt{pN}}\,|g_1|\,|g_2'|+$$
$$+\frac{r}{\sqrt{pN}}\,|g_1'|\,|g_2| \geqslant \frac{|k|}{\sqrt{pN}} \tag{3.10}$$

Since $p>r$, we have $\frac{r}{\sqrt{pN}}|v'| < \sqrt{\frac{r}{N}}|v'|$ and hence using (3.8),

$$\frac{r}{\sqrt{pN}}\,|v'| \in L^2\,[0,\infty). \tag{3.11}$$

Now integrating (3.10) over $(0,\infty)$ and utilising the results (3.8) and (3.11) we see that

$$\int^\infty \frac{|k|}{\sqrt{pN}} \text{ converges,}$$

which is not possible due to the condition given in (3.1).

case (ii) if $r>p$ we divide (3.9) by $\sqrt{rN}$ and utilise the results

$$\sqrt{\frac{r}{N}}\,|v'|,\ \frac{p}{\sqrt{rN}}\,|u'| \in L^2\,[0,\infty)$$

to show that

$$\int^{\infty} \frac{|k|}{\sqrt{rN}} \text{ converges,}$$

contradictory to the condition (3.1).

Thus the assumption $P_{12} \neq 0$ implies that both F_1 and F_2 cannot be square-integrable. Since at least one of $P_{jk} \neq 0$, all the four solutions F_j, $(j=1, 2, 3, 4)$ of the system cannot be square-integrable.

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