Jour. Pure Math.
Vol. I (1981), pp. 81-84

CLASS OF UNIVERSAL ALGEBRAS VALUED ONTO THE SAME UPPER SEMILATTICE

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1. The valuation of a universal algebra has been introduced and studied recently [1, 2, 4]. In fact, it has been defined as a mapping of the universal algebra into an upper semilattice. It has been observed [1] that the lattice of convex subalgebras of a universal algebra is isomorphic to the lattice of convex subsemilattices of the upper semilattice onto which the universal algebra is valued epimorphically.

In the present paper, the authors have studied the class of universal algebras which are valued epimorphically onto the same upper semilattice. It has been shown that these are exactly those universal algebras which have the same upper semilattice of principal convex subalgebras.

2. Let A be a universal algebra with Ω as its domain of operators and let P be an upper semilattice.

A mapping $N : A \rightarrow P$, of A into P, is called a valuation if, and only if,

(i) $N(a_1 \ldots a_n \omega_n) \leqslant N(a_1) \cup \ldots \cup N(a_n)$, where $a_1, \ldots, a_n \in A$; $\omega_n \in \Omega \mid n(\omega_n) = n \geqslant 1$ (n is called the arness of the operator ω_n).

(ii) if $\omega_0 \in \Omega \mid n(\omega_0) = O$, then $N(O_\omega) \leqslant N(a) \;\forall\; a \in A$, where O_ω is the element of A specified by ω_0.

A subset Q of an upper semilattice P is called convex, if, and only if, $\alpha \in Q, x \in P, x \leqslant \alpha \Rightarrow x \in Q$.

A subsemilattice Q of an upper semilattice P is called a convex subsemilattice of P if, and only if, Q is a convex subset of P.

Let $N : A \rightarrow P$, be a valuation of a universal algebra A into an upper semilattice P. A subset B of A is called convex if, and only if,

(1) $b \in B, a \in A, N(a) \leqslant N(b) \Rightarrow a \in B$,

(2) $a, b \in B, N(c) = N(a) \cup N(b), C \in A \Rightarrow c \in B$.

A subalgebra B of A is called convex if, and only if, B is a convex subset of A.

The valuation $N : A \rightarrow P$ is called epimorphic if, and only if, for each $\alpha \in P, \exists\, a \in A \mid N(a) = \alpha$. In this case, we say that the universal algebra A is valued epimorphically onto the upper semilattice P.

3. Let P be an upper semilattice and $\alpha \in P$.

Let $P(\alpha)=\{\beta \in P \mid \beta < \alpha\}$.

Proposition 1. $P(\alpha)$ is a convex subsemilattice of P.

Proof. Indeed, $\beta, \gamma \in P(\alpha) \Rightarrow \beta < \alpha, \gamma < \alpha$.

$\Rightarrow \beta \cup \gamma < \alpha \Rightarrow \beta \cup \gamma \in P(\alpha)$.

Also, $\delta < \gamma ; \gamma \in P(\alpha), \delta \in P$

$\Rightarrow \delta < \gamma < \alpha \Rightarrow \delta \in P(\alpha)$.

Thus, $P(\alpha)$ is a convex subsemilattice of P. $P(\alpha)$ will be called the principal convex subsemilattice of P, generated by α.

Proposition 2. The set L_0^λ (P) of all principal convex subsemilattices of P of P will form an upper semilattice isomorphic to P.

Proof : Let $P(\alpha)$, $P(\beta) \in L_0^\lambda$ (P).

Then $P(\alpha)=\{\gamma \in P \mid \gamma < \alpha\}$ and $P(\beta)=\{\delta \in P \mid \delta < \beta\}$.

Now $P(\alpha \cup \beta)\{\mu \in P \mid \mu < \alpha \cup \beta\}$

Obviously, $P(\alpha), P(\beta) \subseteq P(\alpha \cup \beta)$.

Now let $P(\alpha), P(\beta) \subseteq P(\nu)$.

$\Rightarrow \alpha < \nu, \beta \leqslant \nu \Rightarrow \alpha \cup \beta \leqslant \nu \Rightarrow \alpha \cup \beta \in P(\nu)$

$\Rightarrow P(\alpha \cup \beta) \subseteq P(\nu)$.

Hence $P(\alpha \cup \beta)=P(\alpha) \cup P(\beta)$.

Further, $\alpha \rightarrow P(\alpha)$ is an isomorphism.

4. Let $N : A \rightarrow P$, be a valuation of a universal algebra (A, Ω) into an upper semilattice P.

Let $x \in A$.

Let $A(x)=\{a \in A \mid N(a) < N(x)\}$

Proposition 3 : $A(x)$ is a convex subalgebra of A.

Proof : Indeed, $a_1 \ldots\ldots a_n \in A(x)$, $\omega_n \in \Omega \mid n(\omega_n)=n \geqslant 1$

$\Rightarrow N(a_1 \ldots\ldots a_n \omega_n) \leqslant N(a_1) \cup \ldots\ldots \cup N(a_n) < N(x)$.

$\Rightarrow a_1 \ldots\ldots a_n \omega_n \in A(x)$.

Also $\omega_0 \in \Omega \Rightarrow O_\omega \in A$.

$$\Rightarrow N(O_\omega) \leqslant N(x) - O_\omega \in A(x).$$

Thus, $A(x)$ is a subalgebra of A.

Further, $b \in A(x)$, $a \in A \mid N(a) < N(b)$

$$\Rightarrow N(a) < N(b) \leqslant N(x) - a \in A(x).$$

Also $a, b \in A(x)$, $N(c) = N(a) \cup N(b)$, $c \in A$

$$\Rightarrow N(c) \leqslant N(x) \Rightarrow c \in A(x)$$

Consequently, $A(x)$ is a convex subalgebra of A.

$A(x)$ will be called the principal convex subalgebra of A generated by x.

Proposition 4 : Let N, $A \rightarrow P$ be an epimorphic valuation of the universal A onto the upper semilattice P. Then the set $L^\lambda_o(A)$ of all principal convex subalgebras of A will form an upper semilattice.

Proof : Let $A(x)$, $A(y) \in L^\lambda_o$ (A)

Then $A(x) = \{a \in A \mid N(a) < N(x)\}$, $A(Y) = \{b \in A \mid N(b) < N(y)\}$.

Let $A(x, y) = \{c \in A \mid N(c) < N(x) \cup N(y)\}$

As N is epimorphic, $\exists\, z \in A \mid N(z) = N(x) \cup N(y)$.

Then, by proposition 3, $A(z)$ is a convex subalgebra A.

Evidently $A(x)$, $A(y) \subseteq A(z)$.

Also, if $A(x)$, $A(y) \subseteq A(w)$, then $N(x)$, $N(y) \leqslant N(w)$.

$N(z) = N(x) \cup N(y) < N(w)$.

Thus $A(z)$ is the least upper bound of $A(x)$, and $A(y)$.

5. Let A be a universal algebra with Ω as its domain of operators and P be an upper semilattice and $N : A \rightarrow P$, be an epimorphic valuation of A onto P.

Theorem 1. L^λ_o $(A) \simeq L^\lambda_o$ (P).

Proof : Let $P(\alpha) \in L^\lambda_c$ (P), where $\alpha \in P$.

As N is epimorphic, $\exists\, x \in A \mid N(x) = \alpha$.

Then $A(x) \in L^\lambda_o$ (A).

If $y \in A \mid N(y) = \alpha$, then $A(x) = \{a \in A \mid N(a) < \alpha\} = A(y)$.

Define $P(\alpha)f = A(x)$.

Thus, f is a mapping of L_0^λ (P) into L_0^λ (A).

Evidently, f is a bijection.

Further, $P(\alpha) \subseteq P(\beta)$ in L_0^λ (P)

$\Leftrightarrow \alpha \leqslant \beta$ in P.

$\Leftrightarrow P(\alpha)f \subseteq P(\beta)f$ in L_0^λ (A).

Hence the theorem. From proposition 2 and theorem 1 follows

Theorem 2. Let (A_1, Ω_1), (A_2, Ω_2) be universal algebras and P_1, P_2 be upper semilattices and let $N_1 : A_1 \to P_1$ and $N_2 : A_2 \to P_2$ be epimorphic valuations.

If $L_0^\lambda (A_1) \simeq L_0^\lambda (A_2)$, then $P_1 \simeq P_2$.

Theorems 1 and 2 completely determine the class of universal algebras which are valued epimorphically onto the same upper semilattice.

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Received
26.11.81

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