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# SOME MULTILATERAL GENERATING RELATIONS INVOLVING HERMITE AND TCHEBICHEF POLYNOMIALS

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1. In a recent paper [4], P. A. Lee obtained the following trilateral generating relation involving Charlier and Tchebichef polynomials :

$$(1.1)\quad \sum_{n=0}^{\infty} z^n T_n(x) C_n(k\,;\,\alpha) C_n(l\,;\,\beta)/n\,!$$

$$=\frac{1}{2}\Big[e^{\rho}\Big(1-\frac{\rho}{\alpha}\Big)^k\Big(1-\frac{\rho}{\beta}\Big)^l C_k\Big\{l\,;\,-\frac{(\alpha-\rho)(\beta-\rho)}{\rho}\Big\}$$

$$+e^{\rho'}\Big(1-\frac{\rho'}{\alpha}\Big)^k\Big(1-\frac{\rho'}{\beta}\Big)^l C_k\Big\{l\,;\,-\frac{(\alpha-\rho')(\beta-\rho')}{\rho'}\Big\}\Big],$$

where $\rho=(x+\sqrt{x^2-1})z$, $\rho'=(x-\sqrt{x^2-1})z$,

and $C_n(x\,;\,a)={}_2F_0\Big(-n,\,-x\,;\,-\,;\,-\dfrac{1}{a}\Big)$ ; $a>0$ and $x=0, 1, 2, \ldots\ldots$

The object of the present paper is to derive some multilateral generating relations involving Hermite and Tchebichef polynomials by means of a method which is much shorter than that adopted by Lee. It is interesting to remark that our method of obtaining multilateral generating relations is perfectly general and straightforward in the sense that one can easily apply our method to any $m$-lateral generating relation in order to obtain the corresponding $(m+1)$-lateral generating relation involving Tchebichef polynomial as an extra polynomial. The main results are contained in (2.3), (2.5), (3.5), (3.6) and (3.8).

2. First we consider the following generating relation of W. A. Al-Salam [1] :

$$(2.1)\quad \sum_{n=0}^{\infty} \Delta_{n,\,1,\,2\,;\,x}(H)\,t^n/n\,!=\frac{2x}{(1-t^2)^{3/2}}\exp\Big(\frac{x^2t}{1+t}\Big)$$

where $\Delta_{n,\,1,\,2\,;\,x}(H)=H_{n+1}(x)H_{n+2}(x)-H_n(x)H_{n+3}(x)$

and $\exp\,(xt-\frac{1}{2}t^2)=\sum_{n=0}^{\infty}H_n(x)t^n/n\,!$ .

The following relation of Tchebichef polynomials will be utilized throughout the paper :

$$(2.2)\quad T_n(x)=\tfrac{1}{2}[(x+\sqrt{x^2-1})^n+(x-\sqrt{x^2-1})^n].$$

Instead of using (2.2), Lee used a number of relations (viz. generating relation of $T_n(x)$, successive differentiation, multiplication and Rodrigues' formulas for Laguerre polynomials) in order to prove (1.1).

Now it follows from (2.1) and (2.2) that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \Delta_{n,\,1,\,2\,;\,x}(\mathrm{H})\, T_n(y)$$

$$= \frac{1}{2}\Big[ \sum_{n=0}^{\infty} \frac{\Delta_{n,\,1,\,2\,;\,x}(\mathrm{H})}{n!} \{t(y+\sqrt{y^2-1})\}^n + \sum_{n=0}^{\infty} \frac{\Delta_{n,\,1,\,2\,;\,x}(\mathrm{H})}{n!} \{t(y-\sqrt{y^2-1})\}^n \Big]$$

(2.3)

$$= x\Big[(1-\rho_1^2)^{-3/2} \exp\Big(\frac{x^2\rho_1}{1+\rho_1}\Big) + (1-\rho_2^2)^{-3/2} \exp\Big(\frac{x^2\rho_2}{1+\rho_2}\Big)\Big],$$

where $\rho_1 = t(y+\sqrt{y^2-1})$ and $\rho_2 = t(y-\sqrt{y^2-1})$.

Next we notice that

(2.4) $$\sum_{n=0}^{\infty} T_{n+k}(x) t^n/n!$$

$$= \tfrac{1}{2}[(x+\sqrt{x^2-1})^k \exp\{t(x+\sqrt{x^2-1})\} + (x-\sqrt{x^2-1})^k \exp\{t(x-\sqrt{x^2-1})\}].$$

Thus we have

$$\sum_{n=0}^{\infty} T_n(y)\, t^n/n! \sum_{k=0}^{n} \binom{n}{k} \Delta_{k,\,1,\,2\,;\,x}(\mathrm{H})$$

$$= \sum_{k=0}^{\infty} \frac{\Delta_{k,\,1,\,2\,;\,x}(\mathrm{H})}{k!}\, t^k \sum_{n=0}^{\infty} T_{n+k}(y)\, t^n/n!$$

(2.5)

$$= x\Big[(1-\rho_1^2)^{-3/2} \exp\Big(\rho_1 + \frac{x^2\rho_1}{1+\rho_1}\Big) + (1-\rho_2^2)^{-3/2} \exp\Big(\rho_2 + \frac{x^2\rho_2}{1+\rho_2}\Big)\Big],$$

where $\rho_1 = t(y+\sqrt{y^2-1})$ and $\rho_2 = t(y-\sqrt{y^2-1})$.

**3.** For the sequence of Hermite polynomials $\{H_n(x)\}$, let us suppose

(3.1) $$\sum_{n=0}^{\infty} A_n\, H_{m+n}(x)\, H_n(y)\, t^n = \frac{f(x, y, t)}{[g(x, y, t)]^m} H_m\{h(x, y, t)\},$$

where the sequence of coefficients $A_n$ is selected in such a way that the series on the left of (3.1) gives rise to a generating function separated like the right member of (3.1), $f$, $g$, $h$ being functions of $x$, $y$, $t$ and let

$$(3.2)\quad F(x, t)=\sum_{n=0}^{\infty} a_n H_n(x) t^n$$

be a unilateral generating relation, where $a_n$'s ($\neq 0$) are arbitrary constants.

Then

$$\sum_{n=0}^{\infty} H_n(x) t^n \sum_{k=0}^{n} a_{n-k} A_k H_k(y) z^k$$

$$=\sum_{n=0}^{\infty} \frac{f(x, y, zt)}{[g(x, y, zt)]^n} H_n(h(x, y, zt)) a_n t^n$$

$$=f(x, y, zt)\ F(h(x, y, zt), t/g(x, y, zt))$$

Thus we obtain

$$(3.3)\quad \sum_{n=0}^{\infty} H_n(x) \sigma_n(y, z) t^n = f(x, y, zt)\ F(h(x, y, zt), t/g(x, y, zt))$$

where $\sigma_n(y, z)=\sum_{k=0}^{n} a_{n-k} A_k H_k(y) z^k$.

Now for the existence of a relation like (3.1) we notice that [3] :

$$(3.4)\quad \sum_{n=0}^{\infty} \frac{t^n}{2^n n!} H_{n+m}(x) H_n(y)$$

$$=(1-t^2)^{-\frac{1}{2}(m+1)} \exp\left[\frac{2xyt-(x^2+y^2)t^2}{1-t^2}\right] H_m\left(\frac{x-yt}{(1-t^2)^{1/2}}\right),$$

where $\exp(2xt-t^2)=\sum_{n=0}^{\infty} H_n(x) t^n/n!$, which is slightly different from the symbol $H_n(x)$ used in section 2.

Thus using $A_n=(2^n n!)^{-1}$ in (3.1) we observe that

$$f(x, y, t)=(1-t^2)^{-1/2} \exp\left[\frac{2xyt-(x^2+y^2)t^2}{1-t^2}\right]$$

$$g(x, y, t)=(1-t^2)^{1/2}$$

$$h(x, y, t)=(x-yt)/(1-t^2)^{1/2},$$

so that we derive the trilateral generating relation in the form of the following theorem :

**Theorem 1.** If $F(x, t)=\sum_{n=0}^{\infty} a_n H_n(x) t^n/n\,!$,

where $\exp(2xt-t^2)=\sum_{n=0}^{\infty} H_n(x) t^n/n\,!$,

then

(3.5) $\sum_{n=0}^{\infty} H_n(x)\sigma_n(y, z) t^n/n\,!$

$$=(1-z^2t^2)^{-1/2}\exp\left[\frac{2xyzt-(x^2+y^2)z^2t^2}{1-z^2t^2}\right].F\left(\frac{x-yzt}{(1-z^2t^2)^{1/2}}, \frac{t}{(1-z^2t^2)^{1/2}}\right)$$

where $\sigma_n(y, z)=\sum_{k=0}^{n}\binom{n}{k} a_{n-k}(z/2)^k H_k(y)$, $|zt|<1$.

Now we are in a position to adjoin Tchebichef polynomial to (3.5) in order to derive a quadrilateral generating relation. Indeed we have the following theorem :

**Theorem 2.** If $F(x, t)=\sum_{n=0}^{\infty} a_n H_n(x) t^n/n\,!$

where $\exp(2xt-t^2)=\sum_{n=0}^{\infty} H_n(x) t^n/n\,!$,

then

(3.6) $\sum_{n=0}^{\infty} H_n(x)\sigma_n(y, z) T_n(u) t^n/n\,!$

$$=\frac{1}{2}\sum_{i=1}^{2}\left[(1-z^2\rho_i^2)^{-1/2}\exp\left\{\frac{2xyz\rho_i-(x^2+y^2)z^2\rho_i^2}{1-z^2\rho_i^2}\right\}.F\left(\frac{x-yz\rho_i}{(1-z^2\rho_i^2)^{1/2}}, \frac{\rho_i}{(1-z^2\rho_i^2)^{1/2}}\right)\right],$$

where $\rho_1=t(u+\sqrt{u^2-1})$, $\rho_2=t(u-\sqrt{u^2-1})$,

$$\sigma_n(y, z)=\sum_{k=0}^{n}\binom{n}{k} a_{n-k}(z/2)^k H_k(y),\quad |z\rho_i|<1\,(i=1, 2).$$

As a nice application of theorem 2, we take $a_n=(c)_n$ and use the following divergent generating function due to F. Brafman [2] :

$$(3.7)\quad (1-2xt)^{-c}\ {}_2F_0\left(\tfrac{1}{2}c,\ \tfrac{1}{2}c+\tfrac{1}{2}\ ;\ -\ ;\ \frac{-4t^2}{(1-2xt)^2}\right) \cong \sum_{n=0}^{\infty}(c)_n H_n(x)t^n/n\,!$$

in order to derive at once from Theorem 2 the following relation :

$$(3.8)\quad \sum_{n=0}^{\infty} H_n(x)\sigma_n(y,\ z)T_n(u)t^n/n\,!$$

$$\cong \tfrac{1}{2}\sum_{i=1}^{2}\Big[(1-z^2\rho_i^{\,2})^{c-\frac{1}{2}}(1-z^2\rho_i^{\,2}-2\rho_i x+2yz\rho_i^{\,2})^{-c}.$$

$$.\exp\left(\frac{2xyz\rho_i-(x^2+y^2)z^2\rho_i^{\,2}}{1-z^2\rho_i^{\,2}}\right)\ {}_2F_0\left(\tfrac{1}{2}c,\ \tfrac{1}{2}c+\tfrac{1}{2}\ ;\ -\ ;\ \left(\frac{2\rho_i}{1-z^2\rho_i^{\,2}-2\rho_i x+2yz\rho_i}\right)^2\right)\Big]$$

where $\rho_1=t(u+\sqrt{u^2-1})$, $\rho_2=t(u-\sqrt{u^2-1})$

and $\sigma_n(y,\ z)=\sum_{k=0}^{n}\binom{n}{k}(c)_{n-k}(z/2)^k H_k(y)$.

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