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ON (p+3)-DIMENSIONAL COMPLEX LIE GROUP AND SPECIAL FUNCTIONS

S. K. CHATTERJEA

1. Introduction : In a recent paper [4], C. F. Wong and R. N. Kesarwani have considered the (p+3)-dimensional complex Lie group K_{p+3} with elements

(1.1) $g=g(\tau\,;\,c\,;\,a_0, a_1,\ldots a_p)\,;\ \tau, c, a_i \in C,$

where the multiplication law is given by

(1.2) $g\,(\tau\,;\,c\,;\,a_0, a_1,\ldots a_p)\,g\,(\tau'\,;\,c'\,;\,a'_0, a'_1\ldots a'_p)$
$=g\,(\tau+\tau'\,;\,c+e^{-\tau}c'\,;\,\phi_0, \phi_1,\ldots\phi_p)$

and

$$(1.3)\quad \varphi_l=a_l+\sum_{k=l}^{p}\binom{k}{l}c^{k-l}e^{k\tau}a'_k\,;\ o\leqslant l\leqslant p.$$

Indeed, K_{p+3} is a generalization of 5-dimensional complex Lie group K_5 studied by W. Miller, Jr. [2].

Now the operators $A(g)$, $g \in K_{p+3}$, defining the multiplier representation of K_{p+3} on the complex Vector Space V of all analytic functions $f(z)$ which are defined for all non-zero values of z, are defined by

$$(1.4)\quad [A(g)f](z)=\exp\left(m_0\tau+\mu\sum_{l=0}^{p}a_l z^l\right)(1+c/z)^{w+m_0}f(e^\tau z+e^\tau c),$$

$w+m_0$ being not an integer.

The matrix elements $A_{lk}(g)$ of $A(g)$ with respect to the basis $h_k(z)=z^k$; $k=0, \pm1, \pm2,\ldots$, of V, as defined in [4, p 120] are as follows :

$$(1.5)\quad [A(g)h_k](z)=\sum_{l=-\infty}^{\infty}A_{lk}(g)h_l(z)\,;\ k=0, \pm1, \pm2,\ldots$$

$$\text{i.e.}\quad \exp\left[(m_0+k)\tau+\mu\sum_{l=0}^{p}a_l z^l\right](1+c/z)^{m_0+w+k}\,z^k$$

$$=\sum_{l=-\infty}^{\infty}A_{lk}(g)z^l,\ |\,c/z\,|<1.$$

Wong and Kesarwani have examined one special case of $A_{lk}(g)$, viz. the function ${}_pF_p$, from which ${}_1F_1$ or usual Laguerre function $L_w^{(\alpha)}(z)$ can be easily studied.

The object of the present paper is to examine the general matrix element $A_{lk}(g)$ and then to consider some interesting special cases.

2. General Matrix Element : Let $g=g(\tau\,;\,c\,;\,a_0, a_1,\ldots,a_p)$. Then (1.5) reduces to

$$(2.1)\quad \exp((m_0+k)\tau+\mu a_0)\exp(\mu(a_1z+a_2z^2+\ldots+a_pz^p))(1+c/z)^{m_0+w+k}\, z^k$$

$$=\sum_{l=-\infty}^{\infty} A_{lk}(g)z^l.$$

Expanding the left member of (2.1) with the help of binomial expansion and the following expansion [3, p. 719]

$$(2.2)\quad \exp\,(b_1x+b_2x^2+\ldots)=1+a_1x+a_2x^2+\ldots$$

where

$$a_n=\frac{1}{n\,!}\begin{vmatrix} b_1 & -1 & 0 & \ldots & \ldots & 0 \\ 2b_2 & b_1 & -2 & 0 & \ldots & 0 \\ 3b_3 & 2b_2 & b_1 & -3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}_n$$

$$=\sum\frac{b_1^{e_1}\, b_2^{e_2}\ldots b_n^{e_n}}{e_1\,!\,e_2\,!\ldots e_n\,!}$$

and e_i's are to have all positive integral values inclusive of zero and $e_1+2e_2+3e_3+\ldots+ne_n=n$,

in a power series and computing the coefficient of z^l, we obtain the explicit representation for the matrix element $A_{lk}(g)$ as follows

$$(2.3)\quad A_{lk}(g)=\exp\,((m_0+k)\tau+\mu a_0)\frac{(-c)^{k-l}}{\Gamma(-m_0-w-k)}\sum_{m=0}^{\infty}\frac{\Gamma(m-l-m_0-w)}{\Gamma(k+m-l+1)}\, b_mc^m$$

where

$$(2.4)\quad b_0=1$$

$$(2.5)\quad b_m=\frac{1}{m!}\begin{vmatrix} \mu a_1 & -1 & 0 & 0\ldots & 0 \\ 2\mu a_2 & \mu a_1 & -2 & 0\ldots & 0 \\ 3\mu a_3 & 2\mu a_2 & \mu a_1 & -3\ldots & 0 \\ \ldots & \ldots & \ldots & \ldots\ \ldots & \ldots \end{vmatrix}_m$$

$$=\sum\frac{(\mu a_1)^{e_1}(\mu a_2)^{e_2}\ldots(\mu a_m)^{e_m}}{e_1\,!\;e_2\,!\,\ldots\,e_m\,!}$$

and $e_1+2e_2+3e_3+\ldots+me_m=m.$

If may be noted that there is no simple expression for $A_{lk}(g)$, where $g=g(\tau\,;\,c\,;\,a_0, a_1,\ldots a_p)$, in terms of functions of hypergeometric type.

However, if we put $\beta_m=(-1)^m b_m$, then we obtain the following system of equations :

$$(2.6)\quad \begin{cases} \beta_1-\mu a_1=0 \\ 2\beta_2-2\mu a_2-\beta_1\mu a_1=0 \\ 3\beta_3-3\mu a_3-\beta_1.2\mu a_2-\beta_2\mu a_1=0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ p\beta_p-p\mu a_p-\beta_1(p-1)\mu a_{p-1}-\beta_2(p-2)\mu a_{p-2}-\ldots-\beta_{p-1}\mu a_1=0 \\ (p+1)\beta_{p+1}-\beta_1 p\mu a_p-\beta_2(p-1)\mu a_{p-1}-\ldots-\beta_p\mu a_1=0 \\ (p+2)\beta_{p+2}-\beta_2 p\mu a_p-\ldots-\beta_p.2\mu a_2=0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{cases}$$

By Laurent expansion the coefficients $A_{lk}(g)$ in (2.1) are given by

$$(2.7)\quad A_{lk}(g)$$

$$=\frac{1}{2\pi i}\int^{(0+)}\exp[(m_0+k)\tau+\mu(a_0+a_1z+\ldots+a_pz^p)](1+c/z)^{m_0+w+k}z^{k-l-1}\,dz,$$

where $g=g(\tau\,;\,c\,;\,a_0,a_1,\ldots a_p)$.

Comparing (2.3) with (2.7) we obtain

$$(2.8)\quad \frac{1}{2\pi i}\int^{(0+)}\exp[\mu(a_1z+\ldots+a_pz^p)](1+c/z)^{m_0+w+k}z^{k-l-1}dz$$

$$=\frac{(-c)^{k-l}}{\Gamma(-m_0-w-k)}\sum_{m=0}^{\infty}\frac{\Gamma(m-l-m_0-w)}{\Gamma(k+m-l+1)}c^m\sum\frac{(\mu a_1)^{e_1}(\mu a_2)^{e_2}\ldots(\mu a_m)^{e_m}}{e_1!\,e_2!\ldots e_m!}$$

where $e_1+2e_2+3e_3+\ldots+me_m=m$ and $|c/z|<1$.

3. Special Cases : If in particular, $g=g(\tau\,;\,c\,;\,a,0,\ldots 0,b)$, then from (2.3) we have

$$(3.1)\quad A_{lk}(g)=\exp(m_0+k)\tau+\mu a)\frac{(-c)^{k-l}}{\Gamma(-m_0-w-k)}\sum_{m=0}^{\infty}\frac{\Gamma(m-l-m_0-w)}{\Gamma(k+m-l+1)}\beta_m(-c)^m$$

where

$$\beta_0=1,\ \beta_1=\beta_2=\ldots=\beta_{p-1}=0,\ \beta_p=\mu b,$$

$$\beta_{p+1}=\beta_{p+2}=\ldots=\beta_{2p-1}=0,\ \beta_{2p}=\frac{(\mu b)^2}{2!},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\beta_{kp}=\frac{(\mu b)^k}{k!}.$$

Hence we deduce from (3.1)

$$(3.2)\quad A_{lk}(g)=\exp((m_0+k)\tau+\mu a)\frac{(-c)^{k-l}}{\Gamma(-m_0-w-k)}\sum_{m=0}^{\infty}\frac{\Gamma(mp-l-m_0-w)}{\Gamma(mp+k-l+1)}\cdot[\mu b(-c)^p]^m/m!$$

which has been obtained by Wong and Kesarwani for ${}_pF_p$. This ${}_pF_p$ is a particular case of our matrix element $A_{lk}(g)$ given in (2.3).

Again in particular, if $g=g(0\,;0\,;0, a, b, 0, \ldots 0)$, then the generating function (2.1) becomes

$$(3.3)\quad \exp(\mu(az+bz^2))z^k=\sum_{l=-\infty}^{\infty} A_{lk}(g)z^l.$$

Comparing (3.3) with the generating function of the Hermite polynomials, viz.

$$(3.4)\quad \exp(2xy-y^2)=\sum_{l=0}^{\infty} \frac{y^l}{l!} H_l(x),$$

we obtain

$$(3.5)\quad A_{lk}(g)=\begin{cases} 0 \text{ if } l<k \\ \dfrac{(-\mu b)^{(l-k)/2}}{(l-k)!} H_{l-k}\left(\dfrac{\mu a}{2(-\mu b)^{1/2}}\right), \text{ if } l\geqslant k. \end{cases}$$

Furthermore, in particular, if $g=g(0\,;0\,;0, a, 0, \ldots 0, b)$, then the generating function (2.1) becomes

$$(3.6)\quad \exp(\mu(az+bz^p))z^k=\sum_{l=-\infty}^{\infty} A_{lk}(g)z^l.$$

Comparing (3.6) with the generating function of the polynomials $g_l^p(x)$ considered by L. R. Bragg [1], viz.

$$(3.7)\quad \exp(pzx-z^p)=\sum_{l=0}^{\infty} \frac{z^l}{l!} g_l^{p}(x),$$

we obtain

$$(3.8)\quad A_{lk}(g)=\begin{cases} 0 \text{ if } l<k \\ \dfrac{(-\mu b)^{(l-k)/p}}{(l-k)!} g_{l-k}^p\left(\dfrac{\mu a}{p(-\mu b)^{1/p}}\right), \text{ if } l\geqslant k. \end{cases}$$

When $p=2$, $A_{lk}(g)$ in (3.8) reduces to that in (3.5).

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Dept. of Pure Math.
Calcutta University