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A NOTE ON ROUGH STATISTICAL CONVERGENCE OF ORDER α

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ABSTRACT : In this paper, in the line of Aytar[1] and Çolak [2], we introduce the notion of rough statistical convergence of order α in normed linear spaces and study some properties of the set of all rough statistical limit points of order α.

Key words and phrases : Rough statistical convergence of order α, rough statistical limit points of order α,

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1. INTRODUCTION

The concept of statistical convergence was introduced by Steinhaus [9] and Fast [3] and later it was reintroduced by Schoenberg [8] independently. Over the years a lot of works have been done in this area. The concept of rough statistical convergence of single sequences was first introduced by S. Aytar [1]. Later the concept of statistical convergence of order α was introduced by R. Çolak [2].

If $x = \{x_n\}_{n \in N}$ is a sequence in some normed linear space $(X, \|\cdot\|)$ and r is a nonnegative real number then x is said to rough statistical convergent to $\xi \in X$ if for any $\varepsilon > 0$, $\lim_{n\to\infty} \frac{1}{n}\left|\left\{k \leq n : \|x_k - \xi\| \geq r + \varepsilon\right\}\right| = 0$, [1].

For $r = 0$, rough statistical convergence coincides with statistical convergence.

In this paper following the line of Aytar [1] and Çolak [2] we introduce the notion of rough statistical convergence of order α in normed linear spaces and prove some properties of the set of all rough statistical limit points of order α.

2. BASIC DEFINITIONS AND NOTATONS

Definition 2.1. Let K be a subset of the set of positive integers $\mathbb{N}$. Let $K_n = \{k \in K : k \leq n\}$. Then the natural density of K is given by $\lim_{n\to\infty} \frac{|K_n|}{n}$, where $|K_n|$ denotes the number of elements in K_n.

Definition 2.2. Let K be a subset of the set of positive integers N and α be any real number with $0 < \alpha \leq 1$. Let $K_n = \{k \in K : k \leq n\}$. Then the natural density of order α of K is given by $\lim_{n\to\infty} \frac{|K_n|}{n^\alpha}$, where $|K_n|$ denotes the number of elements in K_n.

Note 2.1. *Let $x = \{x_n\}_{n\in N}$ be a sequence. Then X satisfies some property P for all k except a set whose natural density is zero. Then we say that the sequence x satisfies P for almost all k and we abbreviated this by a.a.k.*

Note 2.2. *Let $x = \{x_n\}_{n\in N}$ be a sequence. Then x satisfies some property P for all k except a set whose natural density of order α is zero. Then we say that the sequence x satisfies P for almost all k and we abbreviated this by a.a.k(α).*

Definition 2.3. Let $x = \{x_n\}_{n\in N}$ be a sequence in a normed linear space $(X, \|\cdot\|)$ and r be a nonnegative real number. Let $0 < \alpha \leq 1$ be given. Then x is said to be rough statistical convergent of order α to $\xi \in X$, denoted by $x_n \xrightarrow{st-r^\alpha} \xi$ *if for any* $\varepsilon > 0$, $\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n : \|x_k - \xi\| \geq r + \varepsilon\}| = 0$, that is a.a.k.($\alpha$) $\|x_k - \xi\| < r + \varepsilon$ for every $\varepsilon > 0$ and some $r > 0$. In this case ξ is called a r^α-st-limit of x.

The set of all rough statistical convergent sequences of order α will be denoted by rS^α for fixed r with $0 < r \leq 1$.

Throughout this paper X will denote a normed linear space and r will denote a nonnegative real number and x will denote the sequence $x = \{x_n\}_{n\in N}$ in X.

In general, the r^α-st-limit point of a sequence may not be unique. So we consider r^α-st-limit set of a sequence x, which is defined by $st\text{-}LIM_x^{r^\alpha} = \{\xi \in X : x_n \xrightarrow{st-r^\alpha} \xi\}$. The sequence x is said to be r^{α}-statistical convergent provided that $st\text{-}LIM_x^{r^{\alpha}} = \emptyset$. For unbounded sequence rough limit set $LIM_x^r = \emptyset$.

But in case of rough statistical convergence of order α $st\text{-}LIM_x^{r^{\alpha}} = \emptyset$ even though the sequence is unbounded. For this we consider the following example.

Example 2.1. *Let $X = \mathbb{R}$. We define a sequence in the following way,*

$$x_n = (-1)^n : i \neq n^2, \alpha = 1$$
$$= n, \text{ otherwise}$$

Then

$$st\text{-}LIM_x^{r^{\alpha}} = \emptyset, \text{ if } r < 1$$
$$= [1 - r, r - 1], \text{ otherwise.}$$

and $LIM_x^{r^{\alpha}} = \emptyset$ for all $r \geq 0$.

Definition 2.4. A sequence $\{x_n\}_{n \in N}$ is said to be statistically bounded if there exists a positive real number M such that $\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : \| x_k \| \geq M\}| = 0$

Definition 2.5. Let $0 < \alpha \leq 1$ be given. A sequence $\{x_n\}_{n \in N}$ is said to be statistically bounded of order α if there exists a positive real number M such that $\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : \| x_k \| \geq M\}| = 0$.

3. MAIN RESULTS

Theorem 3.1. *Let x be a sequence in X. Then x is statistically bounded of order if and only if there exists a nonegative real number r such that* $st\text{-}LIM_x^{r^{\alpha}} = \emptyset$.

Proof. The condition is necessary.

Since the sequence x is statistically bounded, there exists a positive real number M such that $\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \le n : \| x_k \| \ge M\}| = 0$. Let $K = \{k \in \mathbb{N} : \| x_k \| \ge M\}$. Define $r' = \sup\{\| x_k \| : k \in K^c\}$. Then the set $st\text{-}LIM_x^{r'^\alpha}$ contains the origin of X. Hence $st\text{-}LIM_x^{r^\alpha} = \emptyset$.

The condition is sufficient.

Let $st\text{-}LIM_x^{r^\alpha} = \emptyset$ for some $r \ge 0$. Then there exists $l \in X$ such that $l \in st\text{-}LIM_x^{r^\alpha}$. *Then* $\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \le n : \| x_k - l \| \ge r + \varepsilon\}| = 0$ for each $\varepsilon > 0$. Then we say that almost all x_k's are contained in some ball with any radius greater than r. So the sequence x is statistically bounded.

Theorem 3.2. *If* $x' = \{x_{n_k}\}_{n\in \mathbb{N}}$ *is a subsequence of* $x = \{x_n\}_{n\in \mathbb{N}}$ *then* $st\text{-}LIM_x^{r^\alpha} \subseteq st\text{-}LIM_{x'}^{r^\alpha}$.

Proof. The proof is straight forward. So we omit it.

Theorem 3.3. $st\text{-}LIM_x^{r^\alpha}$*, the rough statistical limit set of order* α *of a sequence* x *is closed.*

Proof. If $st\text{-}LIM_x^{r^\alpha} = \emptyset$ then there is nothing to prove. So we assume that $st\text{-}LIM_x^{r^\alpha} = \emptyset$. We can choose a sequence $\{y_k\}_{n\in \mathbb{N}} \subseteq st\text{-}LIM_x^{r^\alpha}$ such that $y_k \to y_*$ for $k \to \infty$. It suffices to prove that $y_* \in st\text{-}LIM_x^{r^\alpha}$.

Let $\varepsilon > 0$. Since $y_k \to y_*$ there exists $k_\varepsilon \in \mathbb{N}$ such that $\| y_k - y_* \| < \frac{\varepsilon}{2}$ for $k > k_\varepsilon$. Now choose $k_0 \in \mathbb{N}$ such that $k_0 > k_\varepsilon$. Then we can write $\| y_{k0} - y_* \| < \frac{\varepsilon}{2}$. Again since $\{y_k\}_{n\in \mathbb{N}} \subseteq st\text{-}LIM_x^{r^\alpha}$ we have $y_{k0} \in st\text{-}LIM_x^{r^\alpha}$. This implies

$$\lim_{n\to\infty n^\alpha} \frac{1}{n^\alpha} \left| \left\{ k \le n : \| x_k - yk_0 \| \ge r + \tfrac{\varepsilon}{2} \right\} \right| = 0 \tag{1}$$

Now we show the inclusion

$$\left\{ k \le n : \| x_k - yk_0 \| < r + \tfrac{\varepsilon}{2} \right\} \subseteq \left\{ k \le n : \| x_k - y_* \| < r + \varepsilon \right\} \tag{2}$$

holds.

Choose $j \in \left\{k \le n : \left\| x_k - y_{k_0} \right\| < r + \frac{\varepsilon}{2}\right\}$. Then we have $\left\| x_j - y_{k_0} \right\| < r + \frac{\varepsilon}{2}$ and hence $\left\| x_j - y_* \right\| \le \left\| x_j - y_{k_0} \right\| + \left\| y_{k_0} - y_* \right\| < r + \varepsilon$ which implies $j \in \{k \le n : \|x_k - y_*\| < r + \varepsilon\}$, which proves the inclusion (2).

From (1) we can say that the set on the right hand side of (2) has natural density 1. Then the set on the left hand side of (2) must have natural density 1. Hence we get $\lim_{n\to\infty} \frac{1}{n^\alpha} \left|\left\{k \le n : \left\| x_k - y_* \right\| \ge r + \varepsilon\right\}\right| = 0$.

This completes the proof. □

Theorem 3.4. *Let x be a sequence in X. Then the rough statistical limit set of order* α $st\text{-}LIM_x^{r^\alpha}$ *is convex.*

Proof : Choose $y_1, y_2 \in st\text{-}LIM_x^{r^\alpha}$ and let $\varepsilon > 0$. Define $K_1 = \{k \le n : \|x_k - y_1\| \ge r + \varepsilon\}$ and $K_2 = \{k \le n : \|x_k - y_2\| \ge r + \varepsilon\}$. Since $y_1, y_2 \in st\text{-}LIM_x^{r^\alpha}$, we have $\lim_{n\to\infty} \frac{1}{n^\alpha}|K_1| = \lim_{n\to\infty} \frac{1}{n^\alpha}|K_2| = 0$. Let λ be any positive real number with $0 \le \lambda \le 1$.

Then

$$\| x_k - [(1 - \lambda)y_1 + \lambda y_2] \| = \| (1 - \lambda)(x_k - y_1) + \lambda(x_k - y_2) \| < r + \varepsilon$$

for each $k \in K_1^c \cap K_2^c$. Since $\lim_{n\to\infty} \frac{1}{n^\alpha}\left| K_1^c \cap K_2^c \right| = 1$, we get

$$\lim_{n\to\infty} \frac{1}{n^\alpha}\left|\left\{k \le n : \left\| x_k - [(1-\lambda)y_1 + \lambda y_2] \right\| \ge r + \varepsilon\right\}\right| = 0$$

that is

$$[(1 - \lambda)y_1 + (\lambda)y_2] \in st\text{-}LIM_x^{r^\alpha}$$

which proves the convexity of the set $st\text{-}LIM_x^{r^\alpha}$. □

Theorem 3.5. *Let x be a sequence in X and $r > 0$. Then the sequence x is rough statistical convergent of order α to $\xi \in X$ if and only if there exists a sequence $y = \{y_n\}_{n\in\mathbb{N}}$ in X such that y is statistically convergent of order α to ξ and $\|x_n - y_n\| \le r$ for all $n \in \mathbb{N}$.*

Proof. The condition is necessary.

Let $x_n \xrightarrow{st-r\alpha} \xi$. Choose any $\varepsilon > 0$. Then we have

$$\lim_{n\to\infty}\frac{1}{n^\alpha}\left|\left\{k \le n : \left\|x_k - \xi\right\| \ge r + \varepsilon\right\}\right| = 0 \text{ for some } r > 0. \tag{3}$$

Now we define

$$y_n = \xi, \text{ if } \|x_n - \xi\| \le r$$

$$= x_n + r\frac{\xi - x_n}{\left\|x_n - \xi\right\|}, \text{ otherwise}$$

Then we can write

$$\| y_n - \xi\| = 0, \text{ if } \| x_n - \xi \| \le r$$

$$= \| x_n - \xi \| - r, \text{ otherwise}$$

and by definition of y_n, we have $\| x_n - y_n \| \le r$ for all $n \in \mathbb{N}$. Hence by (3) and the definition of y_n we get $\lim_{n\to\infty}\frac{1}{n^\alpha}\left|\left\{k \le n : \left\|y_k - \xi\right\| \ge \varepsilon\right\}\right| = 0$. Which implies that the sequence $\{y_n\}_{n\in\mathbb{N}}$ is statistically covergent of order α to ξ.

The condition is sufficient.

Since $\{y_n\}_{n\in\mathbb{N}}$ is statistically convergent of order α to ξ, we have $\lim_{n\to\infty}\frac{1}{n^\alpha}\left|\left\{k \le n : \left\|y_k - \xi\right\| \ge \varepsilon\right\}\right| = 0$ for all $\varepsilon > 0$. Also since for a given $r > 0$ and for the sequence $x = \{x_n\}_{n\in\mathbb{N}}$ $\| x_n - y_n \| < r$, the inclusion $\{k \le n : \| x_k - \xi \| \ge r + \varepsilon\} \subseteq \{k \le n : \| y_k - \xi \| \ge \varepsilon\}$ holds. Hence we get $\lim_{n\to\infty}\frac{1}{n^\alpha}\left|\left\{k \le n : \left\|x_k - \xi\right\| \ge r + \varepsilon\right\}\right| = 0$. This completes the proof.

□

Theorem 3.6. *For an arbitrary* $c \in \Gamma_x$, *where* Γ_x *is the set of all rough statistical cluster points of a sequence* $x \in X$, *we have for a positive real number* r, $\| \xi - c \| \leq r$ *for all* $\xi \in st\text{-}LIM_x^{r,\alpha}$.

Proof. Let $0 < \alpha \leq 1$ be given. On the contrary let assume that there exists a point $c \in \Gamma_x$ and $\xi \in st\text{-}LIM_x^{r,\alpha}$ such that $\| \xi - c \| > r$. Choose $\varepsilon = \dfrac{\|\xi - c\| - r}{3}$. Then

$$\{k \leq n : \| x_k - \xi \| \geq r + \varepsilon\} \supseteq \{k \leq n : \| x_k - c \| < \varepsilon\} \tag{4}$$

holds. Since $c \in \Gamma_x$, we have $\lim\limits_{n\to\infty} \dfrac{1}{n^\alpha}\left|\{k \leq n : \|x_k - c\| < \varepsilon\}\right| \neq 0$. Hence by (4) we have $\lim\limits_{n\to\infty} \dfrac{1}{n^\alpha}\left|\{k \leq n : \|x_k - \xi\| < r + \varepsilon\}\right| \neq 0$. This is a contradictin to the fact $\xi \in st\text{-}LIM_x^{r^\alpha}$.

Theorem 3.7. *Let* x *be sequence in the strictly convex space. Let* r *and* α *be two positive real numbers. If for any* $y_1, y_2 \in st\text{-}LIM_x^{r^\alpha}$ *with* $\| y_1 - y_2 \| = 2r$, *then* x *is statistically convergent of order* α *to* $\dfrac{y_1 + y_2}{2}$.

Proof. Let $z \in \Gamma_x$. Then for any y_1, y_2 $st\text{-}LIM_x^{r^\alpha}$ implies

$$\| y_1 - z \| \leq r \text{ and } \| y_2 - z \| \leq r. \tag{5}$$

On the other hand we have

$$2r = \| y_1 - y_2 \| \leq \| y_1 - z \| + \| y_2 - z \|. \tag{6}$$

Hence by (5) and (6) we get $\| y_1 - z \| = \| y_2 - z \| = r$. Since

$$\frac{1}{2}(y_2 - y_1) = \frac{1}{2}[(z - y_1) + (y_2 - z)] \tag{7}$$

and $\| y_1 - y_2 \| = 2r$, we get $\|\frac{1}{2}(y_2 - y_1)\| = r$. By strict convexity of the space and from the equality (7) we get $\frac{1}{2}(y_1 - y_2) = z - y_1 = y_2 - z$, which implics that $z = \frac{1}{2}(y_1 + y_2)$. Hence z is unique statistical cluster point of the sequence x. On the other hand, from the assumption

$y_1, y_2 \in st\text{-}LIM_x^{r^\alpha}$ implies that $st\text{-}LIM_x^{r^\alpha} \neq \emptyset$. So by Theorem 3.1 the sequence x is statistically bounded of order α. Since z is the unique statistical cluster point of the statistically bounded sequence x of order α we have the sequence x is statistically convergent to $z = \frac{1}{2}(y_1 + y_2)$. □

Theorem 3.8. *Let $0 < \alpha \leq 1$ and x and y be two sequences. Then*

(i) if r^α-st-lim $x = x_0$ and $c \in \mathbb{R}$, then r^α-st-lim $cx = cx_0$

(ii) if r^α-st-lim $x = x_0$ and r^α-st-lim $y = y_0$ then r^α-st-lim$(x + y) = x_0 + y_0$.

Proof. (i) If $c = 0$ it is trivial. Suppose that $c \neq 0$. Then the proof of (i) follows from $\frac{1}{n^\alpha}|\{k \leq n : \| cx_k - cx_0 \| \geq r + \varepsilon\}| \leq \frac{1}{n^\alpha}|\{k \leq n : \|x_k - x_0\| \geq \frac{r+\varepsilon}{|c|}\}| = \frac{1}{n^\alpha}|\{k \leq n : \|x_k - x_0\| \geq \frac{r}{|c|} + \frac{\varepsilon}{|c|}\}|$. Since x is rough statistical convergent of order α, hence cx is also rough statistical convergent of order α.

Again

$$\frac{1}{n^\alpha}|\{k \leq n : \|(x_k + y_k) - (x_0 + y_0)\| \geq r + \varepsilon\}| \leq \frac{1}{n^\alpha}|\{k \leq n : \| x_k - x_0 \| \geq r + \varepsilon\}|$$
$$+ \frac{1}{n^\alpha}|\{k \leq n : \| y_k - y_0 \| \geq r + \varepsilon\}|$$

It is easy to see that every convergent sequence is rough statistical convergent of order α, but the converse is not true always.

Example 3.1. *Let us consider the following sequence of real numbers defined by,*

$$x_k = 1, \text{ if } k = n^3$$
$$= 0, \text{ Otherwise}$$

Then it is easy to see that the sequence is rough statistical convergent of order α with rS^α-lim $x_k = 0$ for $\alpha > \frac{1}{3}$, but it is not a convergent sequence.

Theorem 3.9. *Let $0 < \alpha \leq \beta \leq 1$. Then $rS^{\alpha} \subseteq rS^{\beta}$, where rS^{α} and rS^{β} denote the set of all rough statistical convergent sequence of order α and β respectively.*

Proof. If $0 < \alpha \leq \beta \leq 1$ then

$$\frac{1}{n^{\beta}}|\{k \leq n : \| x_k - l \| \geq r + \varepsilon\}| \leq \frac{1}{n^{\alpha}}|\{k \leq n : \| x_k - l \| \geq r + \varepsilon\}|$$

for every $\varepsilon > 0$ and some $r > 0$ with limit l.

Clearly this shows that $rS^{\alpha} \subseteq rS^{\beta}$.

We do not know whether for a sequence x in X, $r > 0$ and for $0 < \alpha < 1$, $diam\left(st\text{ - }LIM_x^{r^{\alpha}}\right) \leq 2r$ is true or not.

So we leave this above fact as an open problem.

Open Problem 3.1. Is it true for a sequence x in X, $r > 0$ and $0 < \alpha < 1$, $diam\left(st\text{ - }LIM_x^{r^{\alpha}}\right) \leq 2r$.

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