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CHARACTERIZATIONS OF TOPOLOGICAL PROPERTIES VIA STRONG QUASI-UNIFORM COVERS

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ABSTRACT : Pervin [6] demonstrated that every topological space is quasi-uniformizable. It has been observed by Brümmer [1] that unlike the case of uniform covers for a uniform space, a quasi-uniform space cannot be characterized by quasi-uniform covers. In [5], the last two authors introduced the idea of strong quasi-uniform covers, and a characterization of a quasi-uniform space in terms of such covers was proved; moreover; a few topological properties were also formulated in this connection. The purpose of this paper is to continue the study and to characterize a few more topological properties in terms of strong quasi-uniform covers. Furthermore, we study the inter connections among the three—quasi-uniformity, topology and strong quasi-uniform cover, in terms of category.

Keywords : Quasi-uniformity, transitive base (subbase), strong quasi-uniform cover.

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1. INTRODUCTION

In [6] Pervin showed that the development of uniform spaces is a natural one from topological spaces through quasi-uniform spaces. In the same paper he mentioned Császár's [2] assertion that every topological space can be derived from a quasi-uniform space i.e., every topological space is a quasi-uniform space. Now it has been shown [8] that a uniform space can be completely characterized by means of uniform covers. Again, the notion of quasi-uniform covers has been introduced in [3], which is analogous to that of uniform covers. Then intuitively it seems that a quasi-uniform space can be characterized completely by means of quasi-uniform covers. But unfortunately this is not the case. Actually Brümmer pointed this out in [1]. Then in [5], the idea of strong quasi-uniform cover was introduced as a suitable modification of quasi-uniform cover, and quasi-uniform spaces were characterized by such covers. In [5], some topological properties were also characterized by this new type of covers and many other

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topological properties were left out. In this paper some of these remaining topological properties are characterized. Lastly we present quasi-uniformity, topology and strong quasi-uniform cover in terms of category so as to have a better understanding of their interrelations.

2. PREREQUISITES

The concept of quasi-uniformity on a non-empty set was studied in [2] and [3].

Definition 2.1 ([3]). *For a non-empty set X, a subcollection $\mathcal{Q}$ of the power set* $\exp(X \times X)$ *of $X \times X$ is called a quasi-uniformity on X if*

1. $\Delta \subseteq Q$, $\forall Q \in \mathcal{Q}$, *where* $\Delta = \{(x, x) : x \in X\}$.
2. $Q \in \mathcal{Q}$ *and* $Q \subseteq P \subseteq X \times X$ *together imply* $P \in \mathcal{Q}$.
3. *for any two members Q and P of $\mathcal{Q}$,* $\exists R \in \mathcal{Q}$ *such that* $R \subseteq Q \cap P$.
4. *for any member Q of $\mathcal{Q}$,* $\exists P \in \mathcal{Q}$ *such that* $P \circ P \subseteq Q$, *where* $R \circ S = \{(x, z) \in X \times X : \exists y \in X$ *such that* $(x, y) \in R$ *and* $(y, z) \in S\}$, *for* $R, S \in \mathcal{Q}$.

The members of $\mathcal{Q}$ are called entourages of the quasi-uniform space $(X, \mathcal{Q})$.

Definition 2.2 ([3]). *On a non-empty set X, a subcollection $\mathcal{B}$ of a quasi-uniformity $\mathcal{Q}$ is said to from a base for $\mathcal{Q}$ is for each* $Q \in \mathcal{Q}$, $\exists B \in \mathcal{B}$ *such that* $B \subseteq Q$.

Here the quasi-uniformity $\mathcal{Q}$ is said to be the quasi-niformity generated by $\mathcal{B}$.

Definition 2.3 ([3]). *Let $(X, \mathcal{Q})$ be a quasi-uniform space. Then $\mathcal{S} \subseteq \mathcal{Q}$ is called a subbase for $\mathcal{Q}$ if the family of finite intersections of members of $\mathcal{S}$ is a base for $\mathcal{Q}$.*

Here the quasi-uniformity $\mathcal{Q}$ is said to be the quasi-uniformity generated by $\mathcal{S}$.

Theorem 2.4 ([3]). *Let $(X, \mathcal{Q})$ be a quasi-uniform space. Then the collection* $\{G \subseteq X : x \in G \Rightarrow x \in Q(x) \subseteq G$, *for some* $Q \in \mathcal{Q}\}$ *forms a topology on X, where* $Q(x) = \{y \in X : (x, y) \in Q\}$.

This topology is termed as the topology induced by $\mathcal{Q}$ on X and it is denoted by $\tau(\mathcal{Q})$.

In [2] Császár asserted that every topological space can be derived from a quasi-uniform space. This quasi-uniformity for a given topological space is called a compatible quasi-uniformity with the topology, and the given topology is said to admit the quasi-uniformity. In [7], Pervin gave a direct topological construction of a compatible quasi-uniformity for a given topological space and it is called the Pervin quasi-uniformity for the space.

Definition 2.5 ([3]). *A base (subbase) for a quasi-uniformity is said to be transitive if* $B \circ B = B$, *for all members B of the base (subbase), so that each member of the base (subbase) is a transitive relation.*

Definition 2.6 ([3]). *If a quasi-uniformity has a transitive base (or subbase), then it is said to be a transitive quasi-uniformity.*

Example 2.7 ([3]). *Let* (X, τ) *be a topological space. Pervin [7] constructed a quasi-uniformity on* (X, τ) *with the sbbase given by th collection,* $\{T(G, X \setminus G) : G \in \tau\}$, *where* $T(G, X \setminus G)$ *stands for* $(X \times X \setminus (G \times (X \setminus G)))$. *Then the collection of the sets of the form*

$$\bigcap_{i \in \{1,2,\ldots,n\}} T(G_i, X \setminus G_i) = \bigcap_{\Lambda \subseteq \{1,2,\ldots,n\}} ((\bigcap_{i \in \Lambda} G_i \setminus \bigcup_{j \in \{1,2,\ldots,n\} \setminus \Lambda} G_j) \times \bigcap_{i \in \Lambda} G_i) \cup ((X \setminus \bigcup_{i \in \{1,2,\ldots,n\}} G_i) \times X)$$

turn out to be a base for this quasi-uniformity, $\forall n \in \mathbb{N}$. *It is called Pervin's quasi-uniformity. For a topological space, the Pervin's quasi-uniformity is transitive.*

Definition 2.8 ([3]). *For two quasi-uniform spaces* $(X, \mathcal{Q}_X)$ *and* $(Y, \mathcal{Q}_Y)$, *a function* $f : X \to Y$ *is called quasi-uniformly continuous if for each* $Q \in \mathcal{Q}_Y$, $\exists P \in \mathcal{Q}_X$ *such that* $(x_1, x_2) \in P \Rightarrow (f(x_1), f(x_2)) \in Q$.

3. COVERS FOR QUASI-UNIFORM SPACES

We start with the existing notion of quasi-uniform cover for a quasi-uniform space, which is analogous to that of a uniform cover for a uniform space.

Definition 3.1 ([3]). *A cover* C *of a subset A of a quasi-uniform space* $(X, \mathcal{Q})$ *is said to be a quasi-uniform cover of A if* $\exists Q \in \mathcal{Q}$ *such that for each* $a \in A$, $\exists C \in \mathsf{C}$ *with* $a \in Q(a) \subseteq C$.

Example 3.2. *Let* $(X, \mathcal{Q})$ *be a quasi-uniform space and* $\tau = \tau(\mathcal{Q})$ *be the topology induced by* $\mathcal{Q}$ *on X. Then* τ *itself is a quasi-uniform cover of X.*

Now, unlike uniform cover for a uniform space, quasi-uniform cover could not describe quasi-uniformity completely (see [1] for further details). Then in [5], the corresponding authors suitably amended the notion of quasi-uniform cover to have the new one, called strong quasi-uniform cover. In this section first we recall the basics about strong quasi-uniform cover from [5] and then interpret certain characteristics of it to have all the prerequisites for our further discussion. Before proceeding further, we fix here certain notations which will be used afterwards. For a non-empty set X and a non-empty collection $\mathcal{C} \subseteq \exp(X)$, the collection $\{C \in \mathcal{C} : x \in C\}$ will be denoted by $\mathcal{C}_x$, for each $x \in X$. Further by $Q_{\mathcal{C}}$ we will mean the collection $\{(x, y) : x \in X \text{ and } y \in \cap\, \mathcal{C}_x\}$. In [3], $Q_{\mathcal{C}}$ was shown to be a reflexive and transitive relation.

Definition 3.3 ([5]). *Let $(X, \mathcal{Q})$ be a quasi-uniform space and $\mathcal{C}$ be a cover of X. Then $\mathcal{C}$ is said to be a strong quasi-uniform cover of X if $x \in Q(x) \subseteq \cap\, \mathcal{C}_x$, for some $Q \in \mathcal{Q}$ and $\forall x \in X$.*

Now for a quasi-uniform space $(X, \mathcal{Q})$, we immediately have a strong quasi-uniform cover, as the singletone collection $\{X\}$. But it's a trivial one. For non-trivial ones we refer to the following lemma which actually gives us a plenty of strong quasi-uniform covers on a given quasi-uniform space.

Lemma 3.4 ([5]). *For each transitive member Q of a quasi-uniformity $\mathcal{Q}$ on a non-empty set X, $\{Q(x) : x \in X\}$ forms a strong quasi-uniform cover of X.*

It is quite obvious that a strong quasi-uniform cover is a quasi-uniform cover, but not conversely, which follows immediately from Example 3.2.

Theorem 3.5. *Let $\mathcal{B}$ be a transitive base for a compatible quasi-uniformity of a topological space (X, τ). Then $\{B(x) : x \in X, B \in \mathcal{B}\}$ forms an open base for τ.*

Proof. In [3] it has been established that $\{B(x) : B \in \mathcal{B}\}$ is a base for the neighbourhood filter at x, $\forall x \in X$. So we only have to show that $B(x) \in \tau$, $\forall x \in X$ and $\forall B \in \mathcal{B}$. Let $y \in B(x)$ and $z \in B(y)$. So $(x, y), (y, z) \in B \Rightarrow (x, z) \in B \circ B = B \Rightarrow z \in B(x) \Rightarrow B(y) \subseteq B(x)$ i.e., $y \in B(y) \subseteq B(x)$, $\forall y \in B(x)$. Hence $B(x) \in \tau$. The rest of the proof follows immediately.

Theorem 3.6. *Each strong quasi-uniform cover of a quasi-uniform space is an open cover.*

Proof. It follows from the definitions of the strong quasi-uniform cover and the topology generated by a quasi-uniformity.

In [5], the following characterization of a quasi-uniform space was obtained in terms of covers :

Theorem 3.7. *Let $\mathscr{C}$ be the collection of all strong quasi-uniform covrs of a quasi-uniform space $(X, \mathcal{Q})$.* Then

1. *If $\mathsf{C} \in \mathscr{C}$ and D is a cover of X such that $\cap \mathsf{C}_x \subseteq \cap \mathsf{D}_x$ for each $x \in X$, then $\mathsf{D} \in \mathscr{C}$.*
2. *If $\mathsf{C}_1, \mathsf{C}_2 \in \mathscr{C}$, then $\exists \mathsf{C} \in \mathscr{C}$ such that $\cap \mathsf{C}_x \subseteq (\mathsf{C}_1)_x$ and $\cap \mathsf{C}_x \subseteq \cap(\mathsf{C}_2)_x$ for each $x \in X$.*

Conversely, let $\mathscr{C}$ be a collection of covers of a non-empty set X and $\mathscr{C}$ satisfies (1) *and* (2). *Then $\{Q_{\mathsf{C}} : \mathsf{C} \in \mathscr{C}\}$ forms a transitive base for a quasi-uniformity on X and $\mathscr{C}$ is exactly the collection of all strong quasi-uniform covers of this quasi-uniform space.*

Theorem 3.8. *Let $\mathcal{Q}$ be a compatible quasi-uniformity for a topological space (X, τ) and $\mathscr{C}$ be the collection of all strong quasi-uniform covers of X. Then the transitive quasi-uniformity, say $\mathcal{Q}_{\mathscr{C}}$ induced by $\mathscr{C}$ on X is a subcollection of $\mathcal{Q}$ and if $\mathcal{Q}$ is transitive then both are same.*

Proof. Let $\mathsf{C} \in \mathscr{C}$. Then for some $Q \in \mathcal{Q}$, $Q(x) \subseteq \cap \mathsf{C}_x$, $\forall x \in X$. Now $Q \subseteq Q_{\mathsf{C}}$. In fact, $(x, y) \in Q \Rightarrow y \in Q(x) \subseteq \cap \mathsf{C}_x \Rightarrow (x, y) \in Q_{\mathsf{C}}$. Thus $Q_{\mathsf{C}} \in \mathcal{Q}$ and hence $\mathcal{Q}_{\mathscr{C}} \subseteq \mathcal{Q}$.

Now let B be a transitive base for $\mathcal{Q}$, and consider any $B \in \mathsf{B}$. Then by Lemma 3.4, $\mathsf{C} = \{B(x) : x \in X\} \in \mathscr{C}$. We claim that $Q_{\mathsf{C}} \subseteq B$ so that $\mathcal{Q} \subseteq \mathcal{Q}_{\mathscr{C}}$ and we are done. In fact, $(x, y) \in Q_{\mathsf{C}} \Rightarrow y \in \mathsf{C}_x = \cap\{B(z) : x \in B(z), z \in X\} \subseteq B(x)$, as $x \in B(x)$. $\Rightarrow (x, y) \in B$.

From the above result we can infer that distinct quasi-uniformities on a non-empty set may have exactly the same collection of strong quasi-uniform covers.

Theorem 3.9. *Let (X, τ) be a topological space and $\mathcal{Q}$ be a compatible transitive quasi-uniformity for it. The $\{Q_{\mathsf{C}}(x) : x \in X, \mathsf{C} \in \mathscr{C}\}$ i.e., $\{\cap \mathsf{C}_x : x \in X, \mathsf{C} \in \mathscr{C}\}$ is a base for (X, τ), where $\mathscr{C}$ is the collection of all strong quasi-uniform covers of $(X, \mathcal{Q})$.*

Proof. Follows from Theorem 3.5, 3.7 and 3.8 together.

Theorem 3.10. *Let $(X, \mathcal{Q}_X)$ and $(Y, \mathcal{Q}_Y)$ be two quasi-uniform spaces and suppose $\mathcal{Q}_Y$ has a transitive base, say $\mathcal{B}$. Then a function $f : X \to Y$ is quasi-uniformly continuous if and only if given any strong quasi-uniform cover $\mathcal{C}$ of Y, $f^{-1}(\mathcal{C}) = \{f^{-1}(C) : C \in \mathcal{C}\}$ is a strong quasi-uniform cover of X.*

Proof. Let $f: (X, \mathcal{Q}_X) \to (Y, \mathcal{Q}_Y)$ be quasi-uniformly continuous and $\mathcal{C}$ be a strong quasi-uniform cover of Y. Then $\mathcal{C}$ being a strong quasi-uniform cover of Y, $\exists Q \in \mathcal{Q}_Y$ such that $y \in Q(y) \subseteq \cap\mathcal{C}_y$, $\forall y \in Y$. Now f being quasi-uniformly continuous, $\exists P \in \mathcal{Q}_X$ such that $(x_1, x_2) \in P \Rightarrow (f(x_1), f(x_2)) \in Q$. Next let $x \in X$ and $x' \in P(x)$. Also consider $C \in \mathcal{C}$ such that $x \in f^{-1}(C)$. Then $f(x) \in C \Rightarrow f(x) \in Q(f(x)) \subseteq C$. Again $x' \in P(x) \Rightarrow (x, x') \in P \Rightarrow (f(x), f(x')) \in Q \Rightarrow f(x') \in Q(f(x)) \subseteq C$. Since $C \in \mathcal{C}$ is arbitrary, $f(x') \in Q(f(x)) \subseteq \cap\mathcal{C}_{f(x)}$. So, $P(x) \subseteq \cap(f^{-1}(\mathcal{C}))_x$. Thus $f^{-1}(\mathcal{C})$ is a strong quasi-uniform cover of X.

Conversely suppose that the given condition holds and let Q be a member of $\mathcal{Q}_Y$. Then $\mathcal{B}$ being a base for $\mathcal{Q}_Y$, $\exists B \in \mathcal{B}$ such that $B \subseteq Q$. Again by the Theorem 3.4, $\{B(y) : y \in Y\}$ is a strong quasi-uniform cover of Y. Now arguing similarly as in Theorem 3.5, we can show that $y \in B(y) \subseteq \cap\{B(z) : y \in B(z) \text{ and } z \in Y\}$, $\forall y \in Y$. Now by the assumed condition, $\{f^{-1}(B(y)) : y \in Y\}$ is a strong quasi-uniform cover of X. So, there exist $P \in \mathcal{Q}_X$ such that $x \in P(x) \subseteq \cap\{f^{-1}(B(y)) : x \in f^{-1}(B(y)) \text{ and } y \in Y\}$, $\forall x \in X$. Let $(x_1, x_2) \in P$. Then $x_2 \in P(x_1) \subseteq \cap\{f^{-1}(B(y)) : x_1 \in f^{-1}(B(y)) \text{ and } y \in Y\}$. Again $\cap\{f^{-1}(B(y)) : x_1 \in f^{-1}(B(y)) \text{ and } y \in Y\} \subseteq f^{-1}(B(f(x_1)))$. So $x_2 \in f^{-1}(B(f(x_1))) \Rightarrow f(x_2) \in B(f(x_1)) \Rightarrow (f(x_1), f(x_2)) \in B$. Thus f is quasi-uniformly continuous.

The transitivity of $(Y, \mathcal{Q}_Y)$ is not necessary for the necessity part.

4. TOPOLOGICAL PROPERTIES AND STRONG QUASI-UNIFORM COVERS

Here in this section we will characterize some topological properties in terms of strong quasi-uniform covers. In [5], some properties, namely Hausdorffness, compactness, near compactness, paracompactness, near paracompactness, S-closedness, s-closedness, quasi-H-closedness, have been characterized. Here we will address the issues regarding first countability, several separation axioms, separability and connectedness. To that end, we first derive the expressions of closure and intrior of a subset in terms of strong quasi-uniform covers of a quasi-uniform space.

In what follows in this section, it will be assumed (if not stated otherwise), to avoid repetition, that the topology τ of any space (X, τ), under consideration, is generated by a transitive quasi-uniform base $\mathcal{B}$ (in fact, this is always the case in view of Example 2.7).

Lemma 4.1. *Suppose $\mathscr{C}$ is the collection of all strong quasi-uniform covers of a topological space (X, τ). Then for any subset A of X,*

$$cl(A) = \{x \in X : (\cap \mathrm{C}_x) \cap A \neq \phi,\ \forall \mathrm{C} \in \mathscr{C}\}.$$

Proof. Let $x \in cl(A)$ and $\mathrm{C} \in \mathscr{C}$. As $x \in \cap \mathrm{C}_x \in \tau$, we have $(\cap \mathrm{C}_x) \cap A \neq \phi$.

Again, let $x \in X$ such that $(\cap \mathrm{C}_x) \cap A \neq \phi$, $\forall \mathrm{C} \in \mathscr{C}$. Then for any open neighbourhood G of x, $\exists \mathrm{C} \in \mathscr{C}$ such that $x \in \cap\, \mathrm{C}_x \subseteq G$. So $A \cap G \neq \phi$ and hence $x \in cl(A)$.

Lemma 4.2. *If $\mathscr{C}$ denotes the family of all strong quasi-uniform covers of a topological space (X, τ), then for any subset A of X,*

$$int(A) = \{x \in X : \cap \mathrm{C}_x \subseteq A, \textit{ for some } \mathrm{C} \in \mathscr{C}\}.$$

Proof. Let $x \in int(A)$. Then the equality follows immediately by virtue of Theorem 3.9.

Now, let $x \in X$ with $\cap \mathrm{C}_x \subseteq A$ for some $\mathrm{C} \in \mathscr{C}$. Now as $\mathrm{C} \in \mathscr{C}$, $\exists B \in \mathcal{B}$ such that $x \in B(x) \subseteq \cap \mathrm{C}_x$ i.e., $x \in B(x) \subseteq A$. Again $B(x) \in \tau$. Thus $x \in int(A)$.

We are now equipped enough to characterize certain topological concepts via strong quasi-uniform covers.

Theorem 4.3. *Let (X, τ_X) and (Y, τ_Y) be two topological spaces, where τ_X and τ_Y are generated by two quasi-uniformities with transitive bases $\mathcal{B}_X$ and $\mathcal{B}_Y$ respectively. Then a function $f : X \to Y$ is continuous if and only if given $x \in X$ and a strong quasi-uniform cover C_Y of Y, $\exists$ a strong quasi-uniform cover C_X of X such that $f(\cap(\mathrm{C}_X)_x) \subseteq \cap(\mathrm{C}_Y)_{f(x)}$.*

Proof. Let $f : X \to Y$ be continuous and $x \in X$. Again let C_Y be a strong quasi-uniform cover of Y. Then, by Theorem 3.9, $\cap(\mathrm{C}_Y)_{f(x)}$ is an open neighbourhood of $f(x)$. Now by using the same theorem and the continuity of f, we find a strong quasi-uniform cover C_X of X such that $x \in \cap(\mathrm{C}_X)_x$ and $f(\cap(\mathrm{C}_X)_x) \subseteq \cap(\mathrm{C}_Y)_{f(x)}$.

Conversely we assume the given condition and let $x \in X$ and $f(x) \in G \in \tau_Y$. Then $f(x) \in \cap(\mathrm{C}_Y)_{f(x)} \subseteq G$, for some strong quasi-uniform cover C_Y of Y. By hypothesis, there exists a strong quasi-uniform cover C_X of X such that $f(\cap(C_X)_x) \subseteq \cap(C_Y)_{f(x)}$, where $x \in \cap(C_X)_x \in \tau_X$, proving the continuity of f.

Theorem 4.4. *A topological space (X, τ) is T_0 if and only if for each pair of distinct points x and y of X, there exists a strong quasi-uniform cover C of X such that either $y \notin \cap C_x$ or $x \notin \cap C_y$.*

Proof. Let x, y be two distinct points in a T_0-space (X, τ). Then for some open set G of X, either $x \in G$, $y \notin G$ or $x \notin G$, $y \in G$. Then for the first case, there exists a strong quasi-uniform cover C of X such that $x \in \cap C_x \subseteq G$, $y \notin \cap C_x$. Similar is the other case. Thus the result follows.

The converse is also clear in view of Theorem 3.9.

Theorem 4.5. *A topological space (X, τ) is T_1 if and only if for each pair of distinct points x and y of X, there exists a strong quasi-univorm cover C of X such that $y \notin \cap C_x$ and $x \notin \cap C_y$*

Proof. Let x, y be two distinct points in a T_1-space (X, τ). Then there are open sets G_x, G_y in X such that $x \in G_x$, $y \notin G_x$ and $y \in G_y$, $x \notin G_y$. Then there exist strong quasi-uniform covers C_1, C_2 of X such that $x \in \cap (C_1)_x \subseteq G_x$ and $y \in \cap (C_2)_y \subseteq G_y$. Now by Theorem 3.7, there exists a strong quasi-uniform cover C of X such that $x \in \cap C_x \subseteq \cap(C_1)_x$ and $y \in \cap C_y \subseteq \cap (C_2)_y$. This gives $y \notin \cap C_x$ and $x \notin \cap C_y$.

The converse follows from the fact that $\cap C_x, \cap C_y \in \tau$.

Theorem 4.6. *A topological space (X, τ) is T_2 if and only if for any two distinct points x and y of X, there exists a strong quasi-uniform cover C of X such that $(\cap C_x) \cap (\cap C_y) = \phi$.*

Proof. Consider any two distinct points x, y in a T_2-space (X, τ). Then there are two disjoint open sets U, V in X such that $x \in U$, $y \in V$ and $U \cap V = \phi$. Then we have, $x \in \cap(C_1)_x \subseteq U$ and $y \in \cap (C_2)_y \subseteq V$, for some strong quasi-uniform covers C_1, C_2 of X. Now by Theorem 3.7, there exists a strong quasi-uniform cover C of X such that $x \in \cap C_x \subseteq \cap(C_1)_x$ and $y \in \cap C_y \subseteq \cap(C_2)_y$. Since $U \cap V = \phi$, we have $(\cap C_x) \cap (\cap C_y) = \phi$.

Conversely since $\cap C_x, \cap C_y \in \tau(\mathcal{B}) = \tau$, X becomes T_2.

Note 4.7. *Though the above theorem was proved in [5], here we have given a simpler proof.*

The proofs of the next two theorems go along the same line as those of the above three theorems and hence are omitted.

Theorem 4.8. *A topological space (X, τ) is regular if and only if for any given closed set $A \subseteq X$ and $x \in X \setminus A$, there exists a strong quasi-uniform cover C^y of X, for each $y \in A \cup \{x\}$ such that $(\cap C^x{}_x) \cap (\cap C^a{}_a) = \phi$, $\forall a \in A$.*

Theorem 4.9. *A topological space (X, τ) is normal if and only if for any two non-empty closed sets A, $D \subseteq X$ with $A \cap D = \phi$. there exist strong quasi-uniform covers C^x of X, for each $x \in A \cup D$, such that $(\cap C^a{}_a) \cap (\cap C^d{}_d) = \phi$, $\forall a \in A$ and $\forall d \in D$.*

Theorem 4.10. *A topological space (X, τ) is connected if and only if for any non-empty proper subset A of X and any strong quasi-uniform cover C of X, $\exists x \in X$ such that $A \cap (\cap C_x) \neq \phi \neq (X \setminus A) \cap (\cap C_x)$.*

Proof. Let (X, τ) be connected and A be a non-empty proper subset of X. As $Bd(A)$ (=boundary of A) $\neq \phi$, choose $x \in Bd(A)$. Then by Theorem 3.9, Lemma 4.1 and Lemma 4.2, the condition follows immediately.

Conversely suppose that A is a non-empty proper subset of X and the given condition holds for A. then again by Theorem 3.9, Lemma 4.1 and Lemma 4.2, $x \in Bd(A)$ i.e., A has non-empty boundary and so (X, τ) is connected.

Theorem 4.11. *A topological space (X, τ) is first countable if and only if given $x \in X$, $\exists$ a countable family $\{C_n : n \in \mathbb{N}\}$ of strong quasi-uniform covers of X such that $\{\cap (C_n)_x : n \in \mathbb{N}\}$ is a local base at x.*

Proof. Let in a first countable topological space (X, τ), $\{A_n : n \in \mathbb{N}\}$ be a local base at $x \in X$. As $A_n \in \tau$ for each $n \in \mathbb{N}$, there exists a strong quasi-uniform cover C_n of X such that $x \in \cap (C_n)_x \subseteq A_n$, proving that $\{\cap (C_n)_x : n \in \mathbb{N}\}$ is a local base at x.

The converse part is immediate by use of Theorem 3.9.

Theorem 4.12. *A topological space (X, τ) is separable if and only if there exist a countable subset A of X such that for any strong quasi-uniform cover C of X.*

$$(\cap D) \cap A \neq \phi, \ \forall D \subseteq C \text{ with } \cap D \neq \phi.$$

Proof. Let (X, τ) be separable. Then for some countable subset A of X, $cl(A) = X$. Now consider a strong quasi-uniform cover C of X and suppose $\mathsf{D} \subseteq \mathsf{C}$ with $\cap\mathsf{D} \neq \phi$. Also, let $x \in \cap\mathsf{D}$. Now as $x \in cl(A)$, $(\cap\mathsf{C}_x) \cap A \neq \phi$, by Lemma 4.1. Since $\cap\mathsf{C}_x \subseteq \cap\mathsf{D}$, we have $(\cap\mathsf{D}) \cap A \neq \phi$.

Conversely assuming the given condition, let $x \in X$ and $x \in G \in \tau$. Then $x \in \cap\mathsf{C}_x \subseteq G$, for some strong quasi-uniform cover C of X. We need to show that $A \cap G \neq \phi$, which is immediate from the hypothesis that $(\cap\mathsf{C}_x) \cap A \neq \phi$.

5. CATEGORICAL DESCRIPTION OF QUASI-UNIFORMITY

In this section we will describe quasi-uniformity and its interrelations with topology and strong quasi-uniform cover in terms of category. To do this at first we will consider the following definitions and results. All the concepts regarding category used here are taken from [4].

Definition 5.1. *Let X be a non-empty set. A covering structure on X is a collection $\mathscr{C}$ of covers of X such that*

1. *$\mathsf{C} \in \mathscr{C}$ and D is a cover of X such that $\cap\mathsf{C}_x \subseteq \cap\mathsf{D}_x$, for each $x \in X$ implies $\mathsf{D} \in \mathscr{C}$.*
2. *$\mathsf{C}_1, \mathsf{C}_2 \in \mathscr{C}$ implies $\exists \mathsf{C} \in \mathscr{C}$ such that $\cap\mathsf{C}_x \subseteq \cap(\mathsf{C}_1)_x$ and $\cap\mathsf{C}_x \subseteq \cap(\mathsf{C}_2)_x$, for each $x \in X$.*

X together with $\mathscr{C}$, $(X, \mathscr{C})$, is called a covering structure space.

Theorem 5.2. *Let $(X, \mathcal{Q}_X)$ and $(Y, \mathcal{Q}_Y)$ be two quasi-uniform spaces and τ_X and τ_Y be two compatible topologies with $\mathcal{Q}_X$ and $\mathcal{Q}_Y$ respectively. Then the quasi-uniform continuity of $f : (X, \mathcal{Q}_X) \rightarrow (Y, \mathcal{Q}_Y)$ implies the continuity of $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$.*

Proof. The proof is quite straightforward.

Theorem 5.3. *Let (X, τ_X) and (Y, τ_Y) be two topological spaces and $\mathcal{P}_X$ and $\mathcal{P}_Y$ be the Pervin's quasi-uniformities generated by τ_X and τ_Y respectively. Then the continuity of $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$ implies the quasi-uniform continuity of $f : (X, \mathcal{P}_X) \rightarrow (Y, \mathcal{P}_Y)$.*

Proof. The proof is absolutely straightforward, but involves rigorous calculations. Hence we omit it.

Definition 5.4. *Let X and Y be two non-empty sets and $\mathscr{C}_X$ and $\mathscr{C}_Y$ be two collections of covers on X and Y respectively. Then a function $f : (X, \mathscr{C}_X) \to (Y, \mathscr{C}_Y)$ is said to be cover continuous if for any $\mathsf{C} \in \mathscr{C}_Y$, $f^{-1}(\mathsf{C}) = \{f^{-1}(C) : C \in \mathsf{C}\}$ is a member of $\mathscr{C}_X$.*

Theorem 5.5. *Let $(X, \mathcal{Q}_X)$ and $(Y, \mathcal{Q}_Y)$ be two quasi-uniform spaces and $\mathscr{C}_X$ and $\mathscr{C}_Y$ be the collections of strong quasi-uniform covers of X and Y respectively. Then the quasi-uniform continuity of $f : (X, \mathcal{Q}_X) \to (Y, \mathcal{Q}_Y)$ implies the cover continuity of $f : (X, \mathscr{C}_X) \to (Y, \mathscr{C}_Y)$.*

Proof. It follows readily from Theorem 3.10.

Theorem 5.6. *Let $(X, \mathscr{C}_X)$ and $(Y, \mathscr{C}_Y)$ be two covering structure spaces and $\mathcal{Q}_X$ and $\mathcal{Q}_Y$ be the quasi-uniformities generated by $\mathscr{C}_X$ and $\mathscr{C}_Y$ respectively. Then the cover continuity of $f : (X, \mathscr{C}_X) \to (Y, \mathscr{C}_Y)$ implies the quasi-uniform continuity of $f : (X, \mathcal{Q}_X) \to (Y, \mathcal{Q}_Y)$.*

Proof. It again follows from Theorem 3.10.

We now list some major facts about continuous, quasi-uniformly continuous and cover continuous functions to construct certain categories afterwards.

Fact 5.7. 1. *Composition of any two quasi-uniformly continuous (continuous, cover continuous) functions between two quasi-uniform (topological, covering structure) spaces is quasi-uniformly continuous (continuous, cover continuous).*

2. *The identity function on a quasi-uniform (topological, covering structure) space is quasi-uniformly continuous (continuous, cover continuous).*

In view of the above facts we consider the following categories :

1. *QU* : The elements are the quasi-uniform spaces and the morphisms are the quasi-univormly continuous functions among them.
2. *QU** : The elements are the transitive quasi-uniform spaces and the morphisms are the quasi-uniformly continuous functions among them.
3. *Top* : The elements are the topological spaces and the morphisms are the continuous functions among them.
4. *Cov* : The elements are the covering structure spaces and the morphisms are the cover-continuous functions among them.

Theorem 5.8. *QU^* is a full subcategory of QU.*

Proof. It is obvious.

Theorem 5.9. *$F : QU \rightarrow Top$, described by*

$$\begin{array}{ccc} (X,\ \mathcal{Q}_X) & \mapsto & (X,\ \tau_X) \\ f\big\downarrow & & \big\downarrow F(f) = f \\ (Y,\ \mathcal{Q}_Y) & \mapsto & (Y,\ \tau_Y) \end{array}$$

is a faithful, but not a full functor, where $\tau_X = \tau(\mathcal{Q}_X)$, $\tau_Y = \tau(\mathcal{Q}_Y)$.

Proof. The first part follows from Theorem 5.2.

For the last part we consider the real line $\mathbb{R}$. Clearly the uniformity generated by the usual metric d on $\mathbb{R}$ is a quasi-uniformity on $\mathbb{R}$. There are plenty of real-valued continuous functions on $\mathbb{R}$ which are not uniformly continuous. So the functor F cannot be full in general.

Theorem 5.10. *$F^* : QU^* \rightarrow Top$ described by*

$$\begin{array}{ccc} (X,\ \mathcal{Q}_X) & \mapsto & (X,\ \tau_X) \\ f\big\downarrow & & \big\downarrow F(f) = f \\ (Y,\ \mathcal{Q}_Y) & \mapsto & (Y,\ \tau_Y) \end{array}$$

is a faithful, but not a full functor, where $\tau_X = \tau(\mathcal{Q}_X)$, $\tau_Y = \tau(\mathcal{Q}_Y)$.

Proof. The proof is same as that of the previous one.

Theorem 5.11. *$G : Top \rightarrow QU$ described by*

$$\begin{array}{ccc} (X,\ \tau_X) & \mapsto & (X,\ \mathcal{P}(\tau_X)) \\ f\big\downarrow & & \big\downarrow F(f) = f \\ (Y,\ \tau_Y) & \mapsto & (Y,\ \mathcal{P}(\tau_Y)) \end{array}$$

is a fully faithful functor, where $\mathcal{P}(\tau_X)$ and $\mathcal{P}(\tau_Y)$ are the Pervin's quasi-uniformities, generated by τ_X and τ_Y respectively.

Proof. The proof follows from Theorems 5.2 and 5.3.

Theorem 5.12. *$G^* : Top \to QU^*$ described by*

$$\begin{tikzcd}
(X, \tau_X) \arrow[r, mapsto] \arrow[d, "f"'] & (X, \mathcal{P}(\tau_X)) \arrow[d, "F(f) = f"] \\
(Y, \tau_Y) \arrow[r, mapsto] & (Y, \mathcal{P}(\tau_Y))
\end{tikzcd}$$

is a fully faithful functor, where $\mathcal{P}(\tau_X)$ and $\mathcal{P}(\tau_Y)$ are the Pervin's quasi-uniformities, generated by τ_X and τ_Y respectively.

Proof. The proof is similar to that of the above Theorem.

Theorem 5.13. *$I : QU \to Cov$ described by*

$$\begin{tikzcd}
(X, \mathcal{Q}_X) \arrow[r, mapsto] \arrow[d, "f"'] & (X, \mathscr{C}_X) \arrow[d, "F(f) = f"] \\
(Y, \mathcal{Q}_Y) \arrow[r, mapsto] & (Y, \mathscr{C}_Y)
\end{tikzcd}$$

is a fully faithful functor, where $\mathscr{C}_X$ and $\mathscr{C}_Y$ are the collections of strong quasi-uniform covers of X and Y respectively.

Proof. It follows easily from Theorem 3.10.

Theorem 5.14. *$J : Cov \to QU$ described by*

$$\begin{tikzcd}
(X, \mathscr{C}_X) \arrow[r, mapsto] \arrow[d, "f"'] & (X, \mathcal{Q}(\mathscr{C}_X)) \arrow[d, "F(f) = f"] \\
(Y, \mathscr{C}_Y) \arrow[r, mapsto] & (Y, \mathcal{Q}(\mathscr{C}_Y))
\end{tikzcd}$$

is a fully faithful functor, where $\mathcal{Q}(\mathscr{C}_X)$ and $\mathcal{Q}(\mathscr{C}_Y)$ are the quasi-uniformities, generated by $\mathscr{C}_X$ and $\mathscr{C}_Y$ respectively.

Proof. Follows from Theorems 3.7 and 3.10.

Theorem 5.15. *Let us consider the following two categories :*

$I^* : QU^* \to Cov$ described by

$$\begin{tikzcd}
(X, \mathcal{Q}_X) \arrow[r, mapsto] \arrow[d, "f"'] & (X, \mathscr{C}_X) \arrow[d, "F(f) = f"] \\
(Y, \mathcal{Q}_Y) \arrow[r, mapsto] & (Y, \mathscr{C}_Y)
\end{tikzcd}$$

where $\mathscr{C}_X$ and $\mathscr{C}_Y$ are the collections of strong quasi-uniform covers of X and Y respectively.

and

J^* $Cov \to QU^*$ described by

$$\begin{tikzcd}
(X, \mathscr{C}_X) \arrow[r, mapsto] \arrow[d, "f"'] & (X, \mathcal{Q}(\mathscr{C}_X)) \arrow[d, "F(f) = f"] \\
(Y, \mathscr{C}_Y) \arrow[r, mapsto] & (Y, \mathcal{Q}(\mathscr{C}_Y))
\end{tikzcd}$$

where $\mathcal{Q}(\mathscr{C}_X)$ and $\mathcal{Q}(\mathscr{C}_Y)$ are the quasi-uniformities, generated by $\mathscr{C}_X$ and $\mathscr{C}_Y$ respectively.

Then both I^* and J^* are isomorphisms and each is the inverse of the other.

Proof. It follows immediately from Theorems 3.7 and 3.10.

REFERENCES

1. Brümmer, G. C. L.; Functorial transitive quasi-uniformities; *Categorical Topology, Proc. Conf. Toledo, Ohio,* (1983) ©*Heldarmann Verlag, Berlin* (1984); 163–189.
2. *Császár,* Á.; *Fondements de la topologie générale*; Gauthier-Villars, Paris; 1960.
3. Fletcher, P., Lindgren, W.F.; *Quasi-Uniform Spaces;* Marcel Dekker Inc., New York and Basel; 1982.
4. Lane, S.M.; *Categories for the Working Mathematician*; Springer; 1997.

5. Mukherjee, M.N., Debray, A.; Characterization of quasi-uniformity in terms of covers: Some applications; *Ricerche Mat.*; 50(2001); 1–8.

6. Pervin, W.J.; Uniformization of neighbourhood axioms; *Math. Ann.*; 147(1962); 313–315.

7. Pervin, W.J.; Quasi-uniformization of topological spaces; *Math. Ann.;* 147(1962); 316–317.

8. Willard, S.; *General Topoology;* Addison Wesley Publ. Co., Inc.; 1970.

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