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ON SEPARATION AXIOMS WEAKER AND STRONGER THAN REGULARITY AND NORMALITY VIA GRILLS

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ABSTRACT : In this paper, a few types of separation axioms for topological spaces are introduced and studied in terms of grills; of these, one class is contained in and another contains the class of regular spaces. Two other types, one being weaker and another stronger than normality, are also defined and investigated along similar line.

Key words : Grill, topology $\tau_{\mathcal{G}}$, $\mathcal{G}$-g-closed sets, $\mathcal{G}$-g-regular, $\mathcal{G}$-g-normal.

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1. INTRODUCTION AND PRELIMINARIES

Different neighbouring forms of standard separation properties like regularity and normality, are being studied with interest for a long time. Munshi in [7] studied two kinds of separation axioms, called g-regularity and g-normality, stronger than regularity and normality respectively. Our intention in this paper is to follow the idea of Munshi towards introduction of some other separation properties by use of the concept of grills.

In 1947, the idea of grill was first introduced by Choquet [2], a detailed study of which was sbsequently undertaken by Thron [13] and many others. The definition of grill on a topological space X as given by Choquet [2], goes as follows:

Definition 1.1. [2] A nonempty collection $\mathcal{G}$ of nonempty subsets of a topological space X is called a grill if

(i) $A \in \mathcal{G}$ and $A \subseteq B \subseteq X \Rightarrow B \in \mathcal{G}$, and (ii) $A, B \subseteq X$ and $A \cup B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$.

For a grill $\mathcal{G}$ on a topological space (X, τ), Roy and Mukherjee [11] defined an operator Φ from the power set $P(X)$ to $P(X)$ in the following way : For any $A \subseteq X$, , $\Phi(A) = \{x \in X : U \cap A \in \mathcal{G}$ for every open set U containing $x\}$. It was also shown in [11] that the map $\Psi : P(X) \to P(X)$, given by $\Psi(A) = A \cup \Phi(A)$ (for $A \subseteq X$), is a Kuratowski closure operator giving rise to a topology $\tau_{\mathcal{g}}$ (say) on X, finer than τ. Thus a subset A of X is $\tau_{\mathcal{g}}$-closed if $\Psi(A) = A$ or equivalently if $\Phi(A) \subseteq A$. A topological space endowed with a grill $\mathcal{g}$ on X, denote by $(X, \tau, \mathcal{g})$ will be called a grill topological space.

In this paper we introduce and study certain types of separation axioms, termed $\mathcal{g}$-g-regular, g-$\mathcal{g}$-regular, $\mathcal{g}$-g-normal and g-$\mathcal{g}$-normal spaces by using $\mathcal{g}$-g-open sets introduced in [5] and obtain some characterizations of these spaces. Also we obtain some preservation theorems for $\mathcal{g}$-g-regular and $\mathcal{g}$-g-normal spaces in Section 4.

In what follows, by a space X we shall mean a topological space (X, τ). For any $A \subseteq X$, int(A) and cl(A) will respectively stand for the interior and closure of A in (X, τ). Again, $\tau_{\mathcal{g}}$-cl(A) and $\tau_{\mathcal{g}}$-int(A) will respectively mean the closure and interior of A in $(X, \tau_{\mathcal{g}})$. Similarly, whenever we say that a subset A of a space X is open (or closed) in X, these are meant to be so in (X, τ). For open and closed sets with respect to any other topology on X e.g. $\tau_{\mathcal{g}}$ we shall write '$\tau_{\mathcal{g}}$-open' and '$\tau_{\mathcal{g}}$-closed'. A subset A of a space (X, τ) is said to be preopen [6] (α-open [8]) if $A \subseteq \text{int}(\text{cl}(A))$ (resp. $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$). The family of all α-open sets in (X, τ), denoted by τ^{α}, is known to be a topology on X finer than τ, and the closure of A in (X, τ^{α}) is denoted by α-cl(A). We now recall a few definitions and results as prerequisites.

Definition 1.2. A subset A of a space (X, τ) is said to be g-closed [3] (αg-closed [4]) if $\text{cl}(A) \subseteq U$ (resp. $\alpha\text{-cl}(A) \subseteq U$) whenever $A \subseteq U$ and U is open. The complement of a g-closed (αg-closed) set is called a g-open (resp. an αg-open) set.

Definition 1.3.[7] A space (X, τ) is said to be g-regular if for each $x \in X$ and each g-closed set F with $x \notin F$, there exist disjoint open sets U and V such that $x \in U$ and $F \subseteq V$.

Definition 1.4.[7] A space (X, τ) is said to be g-normal if for each pair of disjoint g-closed sets F and K, there exist disjoint open sets U and V such that $F \subseteq U$ and $K \subseteq V$.

Let us now define a space (X, τ) to be αg-normal if for each pair of disjoint αg-closed sets F and K, there exist disjoint open sets U and V such that $F \subseteq U$ and $K \subseteq V$.

Theorem 1.5.[5] Let $\mathcal{g}$ be a grill on a space (X, τ) such that $PO(X)\setminus\{\phi\} \subseteq \mathcal{g}$. Then $\tau_{\mathcal{g}} \subseteq \tau^{\alpha}$, where $PO(X)$ denotes the collection of all preopen sets in (X, τ).

Definition 1.6.[5] Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. Then a subset A of X is said to be g-closed with respect to the grill $\mathcal{G}$ ($\mathcal{G}$-g-closed, for short) if $\Phi(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

A subset A of X is said to be $\mathcal{G}$-g-open if $X \setminus A$ is $\mathcal{G}$-g-closed.

Theorem 1.7.[5] Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. Then for a subset A of X, the following are equivalent:

(a) A is $\mathcal{G}$-g-closed.

(b) $\tau_{\mathcal{G}}$-cl$(A) \subseteq U$ whenever $A \subseteq U$ and U is open.

(c) For all $x \in \tau_{\mathcal{G}}$-cl(A), cl$(\{x\}) \cap A \neq \phi$.

(d) $\tau_{\mathcal{G}}$-cl$(A) \setminus A$ contains no nonempty closed set of (X, τ).

(e) $\Phi(A) \setminus A$ contains no nonempty closed set of (X, τ).

Corresponding to any nonempty subset A of X, a typical grill $[A]$ on X was defined in [12] in the following manner.

Definition 1.8. Let X be a space and $(\phi \neq)\ A \subseteq X$. Then

$$[A] = \{B \subseteq X : A \cap B \neq \phi\}$$

is a grill on X, called the principal grill generated by A.

Remark 1.9. It is shown in [5] that a g-closed set is $\mathcal{G}$-g-closed but not conversely. However, in the case of principal grill $[X]$ generated by X, it is known [12] that $\tau = \tau_{[X]}$, so that any $[X]$-g-closed set becomes simply a g-closed set and vice-versa.

Theorem 1.10.[5] Let (X, τ) be a topological space and $A \subseteq X$. Then $\tau_{\mathcal{G}_\delta} = \tau^\alpha$ and hence a subset A of X is $\mathcal{G}_\delta$-g-closed iff A is αg-closed where $\mathcal{G}_\delta$ is the grill on X given by $\mathcal{G}_\delta = \{A \subseteq X : \text{int}(\text{cl}(A)) \neq \phi\}$.

Theorem 1.11. [5] Let $\mathcal{G}$ be a grill on a space (X, τ). Then $A\ (\subseteq X)$ is $\mathcal{G}$-g-open iff $F \subseteq \tau_{\mathcal{G}}$-int$(A)$ whenever $F \subseteq A$ and F is closed.

Theorem 1.12.[5] For any grill $\mathcal{G}$ on a space (X, τ) the following are equivalent:

(a) Every subset of X is $\mathcal{G}$-g-closed.

(b) Every open subset of (X, τ) is $\tau_{\mathcal{G}}$-closed.

2. $\mathcal{G}$-g-REGULAR AND $\mathcal{G}$-G-REGULAR SPACES

Definition 2.1. Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. Then (X, τ) is said to be $\mathcal{G}$-g-regular if for each $x \in X$ and each $\mathcal{G}$-g-closed set F with $x \notin F$, there exist disjoint open sets U and V such that $x \in U$ and $F \subseteq V$.

Remarks 2.2. Since every closed set is $\mathcal{G}$-g-closed for any grill $\mathcal{G}$ on X, every $\mathcal{G}$-g-regular space is regular. But the converse is false as is shown by the following example.

Example 2.3. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b, c\}, X\}$. Then (X, τ) is regular space but it is not $\mathcal{G}$-g-regular for any grill $\mathcal{G}$ on X. In fact, for any grill $\mathcal{G}$ on X, $F = \{b\}$ is $\mathcal{G}$-g-closed and $c \notin F$, but there are no disjoint open sets which contain c and F.

Remark 2.4. In the case of principal grill $[X]$ generated by X, it is obvious that any $[X]$-g-regular space becomes simply a g-regular space and conversely (refer to Remark 1.9).

Theorem 2.5. Let $\mathcal{G}$ be a grill on a space (X, τ). Then the following are equivalent :

(a) (X, τ) is $\mathcal{G}$-g-regular.

(b) For each $x \in X$ and each $\mathcal{G}$-g-open set U containing x, there exists an open set V in X such that $x \in V \subseteq \text{cl}(V) \subseteq U$.

(c) For each $x \in X$ and each $\mathcal{G}$-g-closed set with $x \notin F$, there exist disjoint open sets U and V such that $x \in U$ and $\tau_{\mathcal{G}}\text{-cl}(F) \subseteq V$.

(d) For each $\mathcal{G}$-g-closed set F and each point $x \in X \setminus F$, there exist open sets U and V of X such that $x \in U$, $F \subseteq V$ and $\text{cl}(U) \cap \text{cl}(V) = \phi$.

Proof. (a) $\Rightarrow$ (*b*) : For a given $x \in X$, let U be any $\mathcal{G}$-g-open set containing x. Then by hypothesis, there exist disjoint open sets V and W such that $x \in V$ and $X \setminus U \subseteq W$. Now $V \cap W = \phi$ $\Rightarrow \text{cl}(V) \subseteq X \setminus W \subseteq U$. Thus $x \in V \subseteq cl(V) \subseteq U$.

(*b*) $\Rightarrow$ (*a*) : Let $x \in X$ and F be a $\mathcal{G}$-g-closed set with $x \notin F$. Then $x \in X \setminus F$ and so by (*a*), there exists an open set V such that $x \in V \subseteq \text{cl}(V) \subseteq X \setminus F$. Put $W = X \setminus \text{cl}(V)$. Then V and W are disjoint open sets such that $x \in V$, $F \subseteq W$. Hence (X, τ) is a $\mathcal{G}$-g-regular space.

(*a*) ⇒ (*c*) : Let $x \in X$ and F be a $\mathcal{G}$-g-closed set not containing x. Then by (*a*), there exist disjoint open sets U and V in X such that $x \in U$ and $F \subseteq V$. Since F is $\mathcal{G}$-g-closed, $\Phi(F) \subseteq V$ i.e., $\tau_{\mathcal{G}}\text{-cl}(F) \subseteq V$.

(*c*) ⇒ (*d*) : Let $x \in X$ and F be a $\mathcal{G}$-g-closed set not containing x. Then by (*c*), there exist disjoint open sets W and V such that $x \in W$ and $\tau_{\mathcal{G}}\text{-cl}(F) \subseteq V$. Also we have $\text{cl}(V) \cap W = \phi$. Now $\text{cl}(V)$ is $\mathcal{G}$-g-closed and $x \notin \text{cl}(V)$. Then again by (*c*), there exist open sets G and H in X such that $x \in G$, $\tau_{\mathcal{G}}\text{-cl}(\text{cl}(V)) \subseteq H$ and $G \cap H = \phi$ i.e., $x \in G$, $\text{cl}(V) \subseteq H$ and $G \cap H = \phi$ and hence $\text{cl}(G) \cap H = \phi$. Now put $U = W \cap G$, then U and V are open subsets of X such that $x \in U$, $F \subseteq V$ and $\text{cl}(U) \cap \text{cl}(V) = \phi$.

(*d*) ⇒ (*a*) : It is clear.

If we take $\mathcal{G} = [X]$ in the above theorem, then by using Remarks 1.9 and 2.4, we have the following result of Noiri and Popa [10].

Corollary 2.6. For the topological space (X, τ), the following are equivalent :

(*a*) (X, τ) is g-regular.

(*b*) For each $x \in X$ and each g-open set U containing x, there exists an open set V in X such that $x \in V \subseteq \text{cl}(V) \subseteq U$.

(*c*) For each $x \in X$ and each g-closed set with $x \notin F$, there exist disjoint open sets U and V such that $x \in U$ and $\text{cl}(F) \subseteq V$.

(*d*) For each g-closed set F and each point $x \in X \setminus F$, there exist open sets U and V of X such that $x \in U$, $F \subseteq V$ and $\text{cl}(U) \cap \text{cl}(V) = \phi$.

We now define another type of separation axiom in grill topological spaces as follows:

Definition 2.7. Let $\mathcal{G}$ be a grill on a space (X, τ). Then (X, τ) is said to be g-$\mathcal{G}$-regular if for each point $x \in X$ and for each closed set F with $x \notin F$, there exist disjoint $\mathcal{G}$-g-open sets U and V such that $x \in U$ and $F \subseteq V$.

Remark 2.8. It is easy to see that the following implication diagram holds :

$\mathcal{G}$-g-regularity ⇒ g-regularity ⇒ regularity ⇒ g-$\mathcal{G}$-regularity.

We show below that a g-$\mathcal{G}$-regular space need not be regular, and hence neither g-regular nor $\mathcal{G}$-g-regular.

Example 2.9. Consider a grill $\mathcal{G} = \{\{b\}, \{a, b\}, \{b, c\}, X\}$ on a space (X, τ) where $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$. It is easy to see that (X, τ) is not regular but it is g-$\mathcal{G}$-regular. In fact, $\Phi(\{a\}) = \phi$, $\Phi(\{a, b\}) = \{b\}$, $\Phi(\{a, c\}) = \phi$. Thus every open set is $\tau_{\mathcal{G}}$-closed and so by Theorem 1.12, every subset of X is $\mathcal{G}$-g-closed and hence every subset of X is $\mathcal{G}$-g-open. Hence (X, τ) is g-$\mathcal{G}$-regular.

Theorem 2.10. Let $\mathcal{G}$ be a grill on a space (X, τ). Then the following are equivalent :

(*a*) X is g-$\mathcal{G}$-regular.

(*b*) For each open set V containing $x \in X$, there exists an open set U such that $x \in U \subseteq \tau_{\mathcal{G}}\text{-cl}(U) \subseteq V$.

Proof. (*a*) ⇒ (*b*) : Let V be any open set in (X, τ) containing a point x of X. Then by hypothesis, there exist disjoint $\mathcal{G}$-g-open sets U and W such that $x \in U$ and $X \setminus V \subseteq W$. Since W is $\mathcal{G}$-g open and $X \setminus V \subseteq W$ with $X \setminus V$ closed, we have by Theorem 1.11, $X \setminus V \subseteq \tau_{\mathcal{G}}\text{-int}(W)$. Now $U \cap W = \phi \Rightarrow U \cap \tau_{\mathcal{G}}\text{-int}(W) = \phi \Rightarrow \tau_{\mathcal{G}}\text{-cl}(U) \subseteq X \setminus \tau_{\mathcal{G}}\text{-int}(W) \subseteq V$. Thus $x \in U \subseteq \tau_{\mathcal{G}}\text{-cl}(U) \subseteq V$.

(*b*) ⇒ (*a*) : Let F be a closed set in X not containing $x \in X$. Then by hypothesis, there exists a $\mathcal{G}$-g-open set U such that $x \in U \subseteq \tau_{\mathcal{G}}\text{-cl}(U) \subseteq X \setminus F$. Put $V = X \setminus \tau_{\mathcal{G}}\text{-cl}(U)$. Then U and V are disjoint $\mathcal{G}$-g-open sets such that $x \in U$ and $F \subseteq V$. Hence (X, τ) is a g-$\mathcal{G}$-regular space.

Let us now recall the following result from [5].

Theorem 2.11. Let $\mathcal{G}$ be a grill on a T_1-space (X, τ) such that $PO(X) \setminus \{\phi\} \subseteq \mathcal{G}$. Then the following are equivalent :

(a) X is regular.

(b) For each closed set F and each $x \in X \setminus F$, there exist disjoint $\mathcal{G}$-g-open sets U and V such that $x \in U$ and $F \subseteq V$.

(c) For each open set V in (X, τ) and each point $x \in V$, there exists a $\mathcal{G}$-g-open set U such that $x \in U \subseteq \tau_{\mathcal{G}}\text{-cl}(U) \subseteq V$.

Combining Theorems 2.10 and 2.11, we get the following result:

Corollary 2.12. Let $\mathcal{G}$ be a grill on a T_1-space (X, τ) such that $PO(X) \setminus \{\phi\} \subseteq \mathcal{G}$. Then (X, τ) is g-$\mathcal{G}$-regular iff it is regular.

3. $\mathcal{G}$-g-NORMAL AND g-$\mathcal{G}$-NORMAL SPACES

As in the last section, we introduce here two variant forms of normality, one being stronger and another weaker than normality.

Definition 3.1. Let $\mathcal{G}$ be a grill on a space (X, τ). Then (X, τ) is said to be $\mathcal{G}$-g-normal if for each pair of disjoint $\mathcal{G}$-g-closed sets F and K, there exist disjoint open sets U and V in X such that $F \subseteq U$ and $K \subseteq V$.

Remark 3.2. Since every closed set is $\mathcal{G}$-g-closed for any grill $\mathcal{G}$ on X, every $\mathcal{G}$-g-normal space in normal. But the converse is false as is shown by the following example.

Example 3.3. Consider a grill $\mathcal{G} = \{\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$ on a topological space (X, τ) where $X = \{a, b, c, d\}$, $\tau = \{\phi, \{b\}, X\}$. Then (X, τ) is normal but is not $\mathcal{G}$-g-normal. In fact, every open subset of X is $\tau_{\mathcal{G}}$-closed and hence by Theorem 1.12 every subset of X is $\mathcal{G}$-g-closed. Now $F = \{a, d\}$, $K = \{b, c\}$ are disjoint $\mathcal{G}$-g-closed sets, but they cannot be separated by disjoint open sets in X.

Theorem 3.4. Let $\mathcal{G}$ be a grill on a space (X, τ). Then (X, τ) is $\mathcal{G}$-g-normal iff for each $\mathcal{G}$-g-closed set F and each $\mathcal{G}$-g-open set U containing F, there exists an open set V in X such that $F \subseteq V \subseteq \text{cl}(V) \subseteq U$.

Proof. The straightforward proof is omitted.

If the principal grill $[X]$ takes the role of $\mathcal{G}$ in the above theorem, then we obtain the following characterizations of a g-normal space.

Corollary 3.5. A topological space (X, τ) is g-normal iff for each g-closed set F and for any g-open set U containing F, there exists an open set V of X such that $F \subseteq V \subseteq \mathrm{cl}(V) \subseteq U$.

Theorem 3.6. The following are equivalent for a grill topological space $(X, \tau, \mathcal{G})$:

(*a*) X is $\mathcal{G}$-g-normal.

(*b*) For each pair of disjoint $\mathcal{G}$-g-closed sets F and K, there exists an open set U in X containing F such that $\mathrm{cl}(U) \cap K = \phi$.

(*c*) For each pair of disjoint $\mathcal{G}$-g-closed sets F and K, there exist open sets U and V in X such that $F \subseteq U$, $K \subseteq V$ and $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \phi$.

Proof. (*a*) $\Rightarrow$ (*b*) : Let F and K be two disjoint $\mathcal{G}$-g-closed sets. Then by Theorem 3.4, there exists an open set U such that $F \subseteq U \subseteq \mathrm{cl}(U) \subseteq X \setminus K$. Thus for the open set U we have, $F \subseteq U$ and $\mathrm{cl}(U) \cap K = \phi$.

(*b*) $\Rightarrow$ (*c*) : Let F and K be two disjoint $\mathcal{G}$-g-closed sets. Then by (*b*), there exists an open set U in X such that $F \subseteq U$ and $\mathrm{cl}(U) \cap K = \phi$. Again, since K and $\mathrm{cl}(U)$ are disjoint $\mathcal{G}$-g-closed sets, by hypothesis there exists an open set V such that $K \subseteq V$ and $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \phi$.

(*c*) $\Rightarrow$ (*a*) : Obvious.

If in the above theorem, we take $\mathcal{G} = [X]$, then by using Remark 1.9, we arrive at the following known result (viz Theorem 4.1 of Noiri and Popa [10]).

Corollary 3.7. For a topological space X, the following are equivalent :

(*a*) X is g-normal.

(*b*) For each pair of disjoint g-closed sets F and K, there exists an open set U in X containing F such that $\mathrm{cl}(U) \cap K = \phi$.

(*c*) For each pair of disjoint g-closed sets F and K, there exist open sets U and V in X such that $F \subseteq U$, $K \subseteq V$ and $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \phi$.

If we set $\mathcal{G} = \mathcal{G}_\delta$ in Theorem 3.6, we get the following result.

Corollary 3.8. For a topological space X, the following are equivalent :

(*a*) X is αg-normal.

(*b*) For each pair of disjoint αg-closed sets F and K, there exists an open set U in X containing F such that $\mathrm{cl}(U) \cap K = \phi$.

(*c*) For each pair of disjoint αg-closed sets F and K, there exist open sets U and V in X such that $F \subseteq U$, $K \subseteq V$ and $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \phi$.

Theorem 3.9. Let $\mathcal{G}$ be a grill on a space (X, τ). If F and K are disjoint $\mathcal{G}$-g-closed sets of a $\mathcal{G}$-g-normal space (X, τ), then there exist disjoint open sets U and V in X such that $\tau_{\mathcal{G}}$-cl$(F) \subseteq U$, $\tau_{\mathcal{G}}$-cl$(K) \subseteq V$.

Proof. Let F and K be two disjoint $\mathcal{G}$-g-closed sets of a $\mathcal{G}$-g-normal space (X, τ). Then there exist disjoint open sets U and V such that $F \subseteq U$ and $K \subseteq V$. Since F is $\mathcal{G}$-g-closed and $F \subseteq U$, $\tau_{\mathcal{G}}$-cl$(F) \subseteq U$. Similarly, $\tau_{\mathcal{G}}$-cl$(K) \subseteq V$.

Theorem 3.10. Let $\mathcal{G}$ be a grill on a $\mathcal{G}$-g-normal space (X, τ). If F is a $\mathcal{G}$-g-closed set and V be a $\mathcal{G}$-g-open set in X such that $F \subseteq V$, then there exists an open set U in X such that $F \subseteq \tau_{\mathcal{G}}$-cl$(F) \subseteq U \subseteq \tau_{\mathcal{G}}$-int$(V) \subseteq V$.

Proof. Since F and $X \setminus V$ are disjoint $\mathcal{G}$-g-closed sets of a $\mathcal{G}$-g -normal space (X, τ), by Theorem 3.9, there exist disjoint open sets U and W such that $\tau_{\mathcal{G}}$-cl$(F) \subseteq U$ and $\tau_{\mathcal{G}}$-cl$(X \setminus V) \subseteq W$. Now $U \subseteq X \setminus W \subseteq \tau_{\mathcal{G}}$-int$(V) \subseteq V$. Thus $F \subseteq \tau_{\mathcal{G}}$-cl$(F) \subseteq U \subseteq \tau_{\mathcal{G}}$-int$(V) \subseteq V$.

If we set $\mathcal{G} = [X]$ in the above theorem, we have the following result.

Corollary 3.11. Let (X, τ) be a g-normal space. If F is a g-closed set and V a g-open set containing F, then there exists an open set U in X such that $F \subseteq \mathrm{cl}(F) \subseteq U \subseteq \mathrm{int}(V) \subseteq V$.

Definition 3.12. Let $\mathcal{G}$ be a grill on a space (X, τ). Then (X, τ) is said to be g-$\mathcal{G}$-normal if for each pair of disjoint closed sets F and K, there exist disjoint $\mathcal{G}$-g-open sets U and V such that $F \subseteq U$ and $K \subseteq V$.

Remark 3.13. Since every open set is $\mathcal{G}$-g-open for any grill $\mathcal{G}$ on X, every normal space is $\mathcal{G}$-g-normal. That the converse is false is shown by the following example.

Example 3.14. Consider the grill $\mathcal{G} = \{\{b\}, \{a, b\}, \{b, c\}, X\}$ on the topological space (X, τ) where $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$. Then (X, τ) is clearly not normal. But

(X, τ) is g-$\mathcal{G}$-normal. In fact, $\Phi(\{a\}) = \phi$, $\Phi(\{a, b\}) = \{b\}$ and $\Phi(\{a, c\}) = \phi$. Thus every open set of X is $\tau_{\mathcal{G}}$-closed and hence by Theorem 1.12, every subset of X is $\mathcal{G}$-g-closed and so every subset of X is $\mathcal{G}$-g-open.

Theorem 3.15. Let $\mathcal{G}$ be a grill on a space (X, τ). Then the following are equivalent:

(*a*) X is g-$\mathcal{G}$-normal.

(*b*) For each closed set F and each open set V containing F, there exists a $\mathcal{G}$-g-open set U such that $F \subseteq U \subseteq \tau_{\mathcal{G}}\text{-cl}(U) \subseteq V$.

Proof. (*a*) $\Rightarrow$ (*b*) : Let F be a closed set and V be an open set such that $F \subseteq V$. Then by (*a*), there exist disjoint $\mathcal{G}$-g-open sets U and W such that $F \subseteq U$ and $X \setminus V \subseteq W$. Now $U \cap W = \phi \Rightarrow U \cap \tau_{\mathcal{G}}\text{-int}(W) = \phi \Rightarrow \tau_{\mathcal{G}}\text{-cl}(U) \subseteq X \setminus \tau_{\mathcal{G}}\text{-int}(W)$. Since $X \setminus V \subseteq W$ where W is $\mathcal{G}$-g-open, by Theorem 1.11, $X \setminus V \subseteq \tau_{\mathcal{G}}\text{-int}(W)$. Thus $F \subseteq U \subseteq \tau_{\mathcal{G}}\text{-cl}(U) \subseteq X \setminus \tau_{\mathcal{G}}\text{-int}(W) \subseteq V$.

(*b*) $\Rightarrow$ (*a*) : Let F and K be any two disjoint closed subsets of X. Then by (*b*), there exists a $\mathcal{G}$-g-open set U such that $F \subseteq U \subseteq \tau_{\mathcal{G}}\text{-cl}(U) \subseteq X \setminus K$. Put $V = X \setminus \tau_{\mathcal{G}}\text{-cl}(U)$. Then U and V are disjoint $\mathcal{G}$-g-open sets such that $F \subseteq U$ and $K \subseteq V$. Hence (X, τ) is g-$\mathcal{G}$-normal.

In [5], the following theorem was deduced:

Theorem 3.16. Let $\mathcal{G}$ be such a grill on a space (X, τ) that $PO(X)\setminus\{\phi\} \subseteq \mathcal{G}$. Then the following are equivalent:

(a) X is normal.

(b) For each pair of disjoint closed sets F and K, there exist disjoint $\mathcal{G}$-g-open sets U and V such that $F \subseteq U$ and $K \subseteq V$.

(c) For each closed set F and any open set V containing F, there is a $\mathcal{G}$-g-open set U such that $F \subseteq U \subseteq \tau_{\mathcal{G}}\text{-cl}(U) \subseteq V$.

Now from Theorem 3.15 and Theorem 3.16, we get the following result :

Corollary 3.17. Let $\mathcal{G}$ be a grill on a space (X, τ) such that $PO(X) \setminus \{\phi\} \subseteq \mathcal{G}$. Then (X, τ) is g-$\mathcal{G}$-normal iff (X, τ) is normal.

Theorem 3.18. Let $\mathcal{G}$ be a grill on a space (X, τ). Let F be closed and K be g-closed in a

g-$\mathcal{G}$-normal space (X, τ) such that $F \cap K = \phi$. Then there exist disjoint $\mathcal{G}$-g-open sets U and V such that $\text{cl}(K) \subseteq U$ and $F \subseteq V$.

Proof. Let F be closed and K be a g-closed set in a $\mathcal{G}$-g-normal space X such that $F \cap K = \phi$. Since K is g-closed, $\text{cl}(K) \subseteq X \setminus F$, again since X is g-$\mathcal{G}$-normal and $\text{cl}(K) \cap F = \phi$, there exist disjoint $\mathcal{G}$-g-open sets U and V such that $\text{cl}(K) \subseteq U$ and $F \subseteq V$.

4. GENERALIZED CONTINUOUS FUNCTIONS VIA GRILL

We begin this section by quoting the following definition from [1]:

Definition 4.1. A function $f : X \to Y$ is said to be generalized continuous (g-continuous, for short) if the inverse image of every closed set in Y is g-closed in X.

In an analogous manner, we now define s follows:

Definition 4.2. A function $f : X \to Y$ is said to be generalized continuous with respect to some grill $\mathcal{G}$ on X ($\mathcal{G}$-g-continuous, for short) if the inverse image of every closed set in Y is $\mathcal{G}$-g-closed in X.

Observation 4.3. A function $f : X \to Y$ is $\mathcal{G}$-g continuous with respect to some grill $\mathcal{G}$ on X iff the inverse image of every open set in Y is $\mathcal{G}$-g-open in X.

Remark 4.4.

(i) It is shown in [1] that every continuous function is g-continuous and but not conversely.

(ii) It is now clear that every g-continuous function is $\mathcal{G}$-g-continuous; but the converse is false. In fact, let $X = \{a, b, c\}$, $\tau = \{\phi, \{c\}, \{b, c\}, X\}$ and $\mathcal{G} = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$; $Y = \{x, y\}$, $\sigma = \{\phi, \{y\}, Y\}$.

We define a function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ as follows:

$$f(a) = f(b) = y,\ f(c) = x.$$

Then f is $\mathcal{G}$-g-continuous but it is not g-continuous. In fact, $F = \{x\}$ is closed in (Y, σ), $f^{-1}(F) = \{c\}$ is $\mathcal{G}$-g-closed in $(X, \tau, \mathcal{G})$ but it is not g-closed.

Definition 4.5. A function $f : (X, \tau, \mathcal{G}_1) \to (Y, \sigma, \mathcal{G}_2)$ is said to be $(\tau\mathcal{G}_1, \tau\mathcal{G}_2)$-closed if the image of every $\tau\mathcal{G}_1$-closed set in X is $\sigma\mathcal{G}_2$-closed in Y.

From now on, whenever $\mathcal{G}_1$(resp. $\mathcal{G}_2$) is a grill on a space X (resp. on Y) and $A \subseteq X$ (resp. $\subseteq Y$) is generalized closed or open with respect to $\mathcal{G}_1$ (resp. $\mathcal{G}_2$), we shall write, to simplify notation, that A is $\mathcal{G}$-g-closed or $\mathcal{G}$-g-open in X (resp. Y) and hope that the context will make things clear.

Theorem 4.6. Let A ($\subseteq X$) be a $\mathcal{G}$-g-closed set in a grill topological space $(X, \tau, \mathcal{G}_1)$. If $f : (X, \tau, \mathcal{G}_1) \to (Y, \sigma, \mathcal{G}_2)$ is continuous and $(\tau\mathcal{G}_1, \sigma\mathcal{G}_2)$-closed, then $f(A)$ is a $\mathcal{G}$-g-closed set in $(Y, \sigma, \mathcal{G}_2)$.

Proof. Let $f(A) \subseteq U$ where U is open in Y. Then $A \subseteq f^{-1}(U)$ and $f^{-1}(U)$ is open in X. Since A is $\mathcal{G}$-g-closed, $\Phi(A) \subseteq f^{-1}(U)$ so that $A \cup \Phi(A) \subseteq f^{-1}(U)$. Thus $\tau\mathcal{G}_1$-cl$(A) \subseteq f^{-1}(U)$ and hence $f(\tau\mathcal{G}_1$-cl$(A) \subseteq U$. Since f is $(\tau\mathcal{G}_1, \sigma\mathcal{G}_2)$-closed, $f(\tau\mathcal{G}_1$-cl$(A))$ is $\sigma\mathcal{G}_2$-closed and $\sigma\mathcal{G}_2$-cl$(f(A)) = f(\tau\mathcal{G}_1$-cl$(A) \subseteq U$.Hence $f(A)$ is $\mathcal{G}$-g-closed in Y.

However the image of a $\mathcal{G}$-g-closed set need not be $\mathcal{G}$-g-closed under continuous function as is shown below.

Example 4.7. Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $\mathcal{G}_1 = \{\{c\}, \{a, c\}, \{b, c\}, X\}$; $Y = \{x, y, z\}$, $\sigma = \{\phi, \{x\}, \{y\}, \{x, y\}, Y\}$, $\mathcal{G}_2 = \{\{x\}, \{y\}, \{x, y\}, \{x, z\}, \{y, z\}, Y\}$.

We define a function $f : (X, \tau, \mathcal{G}_1) \to (Y, \sigma, \mathcal{G}_2)$ as follows :

$$f(a) = x,\ f(b) = f(c) = y.$$

Then f is continuous. Now $A = \{a, c\}$ is $\mathcal{G}$-g-closed but its image $f(A) = \{x, y\}$ is not $\mathcal{G}$-g-closed.

Definition 4.8. A function $f : X \to Y$, where X, Y are two grill topological spaces, is said to be irresolute with respect to some grill $\mathcal{G}$ on X ($\mathcal{G}$-g-irresolute, for short) if the inverse image of every $\mathcal{G}$-g-open set in Y is $\mathcal{G}$-g-open in X.

Theorem 4.9. Let $f : X \to Y$ be open, $\mathcal{G}$-g-irresolute and surjective. Then

(a) X is $\mathcal{G}$-g-regular $\Rightarrow$ Y is $\mathcal{G}$-g-regular.

(b) X is $\mathcal{G}$-g-normal $\Rightarrow$ Y is $\mathcal{G}$-g-normal.

Proof. (a) Let F be $\mathcal{G}$-g-closed in Y and $y \in Y \setminus F$. Since f is $\mathcal{G}$-g-irresolute, $f^{-1}(F)$ is $\mathcal{G}$-g-closed in X. Also $x \notin f^{-1}(F)$ where $x \in f^{-1}(y)$. Since X is $\mathcal{g}$-g-regular, there exist disjoint open sets U and V in X such that $x \in U$ and $f^{-1}(F) \subseteq V$. Since f is open and surjective, $f(U)$ and $f(V)$ are disjoint open sets in Y such that $y \in f(U)$ and $F \subseteq f(V)$. Hence Y is $\mathcal{g}$-g-regular.

(*b*) The proof is quite similar to that of (a) and hence is omitted.

Theorem 4.10. Let $f : (X, \tau, \mathcal{g}_1) \to (Y, \sigma, \mathcal{g}_2)$ be continuous and a $(\tau\mathcal{g}_1, \sigma\mathcal{g}_2)$-closed injective mapping.

(*a*) If Y be $\mathcal{g}$-g-regular then X is $\mathcal{g}$-g-regular.

(*b*) If Y be $\mathcal{g}$-g-normal then X is $\mathcal{g}$-g-normal.

Proof. (*a*) Let F be $\mathcal{g}$-g-closed in X and $x \in X \setminus F$. Since f is continuous and $(\tau\mathcal{g}_1, \sigma\mathcal{g}_2)$-closed, by Theorem 4.6. $f(F)$ is $\mathcal{g}$-g-closed in Y and also $f(x) \notin f(F)$. Again since Y is $\mathcal{g}$-g-regular, there exist disjoint open sets U and V such that $f(x) \in U$ and $f(F) \subseteq V$. Thus $x \in f^{-1}(U)$ and $F \subseteq f^{-1}(V)$. Since f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are open in X and hence X is $\mathcal{g}$-g-regular.

The proof of (b) is quite similar and left.

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