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SOME IDENTITIES INVOLVING SUMS OF LUCAS NUMBERS

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ABSTRACT : Several combinatorial identities involving Lucas numbers. Fibonacci numbers and exponents of two are derived by the mathematical induction. The notion of the Lucas matrix $\mathcal{L}_n^{(s)}$ of type s, which contains Lucas numbers along the main diagonal parallels, is introduced. Regular case $s = 0$ is contained in the results obtained in [19]. In the present paper the case $s = 1$ is investigated. The inverse of the Lucas matrix $\mathcal{L}_n^{(1)}$ is derived using previously defind identities. Factorization of the general Pascal matrix as well as a particular factorization of the block-Pascal matrix in terms of the matrix $\mathcal{L}_n^{(1)}$ is given. Additional combinatorial identities referring to the generalized Fibonacci numbers, binomial coefficients and special functions are derived as implications of these matrix correlations.

Key words : Matrix inverse; Fibonacci numbers; Lucas numbers; Fibonacci matrix: Pascal matrix; Lucas matrix; Toeplitz matrix.

AMS Subj. Class. : 05A10, 05A19, 15A09, 11B39, 11B68.

1. INTRODUCTION

The Fibonacci numbers $\{F_n\}_{n=0}^{\infty}$ are terms of the sequence satisfying initial conditions $F_0 = 0$, $F_1 = 1$, and the recurrence relation $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$. Close companions to the Fibonacci numbers, are the Lucas numbers $\{L_n\}_{n=0}^{\infty}$, which follow the same recursive pattern, but begin with $L_0 = 2$ and $L_1 = 1$. The Fibonacci and Lucas sequences has been discussed in many articles and books (sec. for example [9]). Fibonacci and Lucas numbers have long interested mathematicians for theoretical development and applications. For example, an application of these numbers in graph theory has been investigated in [21], while an interesting

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application of Fibonacci numbers in codding theory has been studied in [17]. The ratio of two consecutive of these numbers converges to the Golden section $\alpha = (1 + \sqrt{5})/2$, which application appears in many research areas, particularly in Physics. Engineering, Architecture, Nature and Art. Naschie and Marek-Crnjac gave some examples of the Golden ratio in Theoretical Physics and Physics of High Energy Particles [13, 14, 15]. A few mathematical concepts of similarity and proportion are known to be critical in understanding the growth processes in the natural world [23]. For example, the Fibonacci series is well known to lie at the heart of plant growth and living organisms [10]. The Lucas relationship within the 20 canonical amino acids has been shown in [23].

In the present paper we derive some additional combinatorial identities expressing sums which include Lucas numbers. Our results are derived using lower triangular Toeplitz matrices (Toeplitz matrices are constant along the diagonals) containing Lucas numbers.

The outline of this paper is as follows. The explicit representation of the inverse of the matrix $\mathcal{L}_n^{(1)}$ is derived in the second section using auxiliary combinatorial identity referring to Lucas numbers. A factorization of the generalized Pascal matrix in terms of the Lucas matrix $\mathcal{L}_n^{(1)}$ are considered in Section 3. Some combinatorial identities involving binomial coefficients, the Lucas numbers and special functions are derived as corollaries in the third section. In the fourth section we generalize principles used in the the papers [12, 19, 18, 26] as well as in the previous section to a more general class of Pascal-like matrices. For this purpose, we define generalized block-Pascal matrices of type s as a generalization of generalized Pascal matrices. Factorization of the block-Pascal matrix of type 2 with respect to the matrix $\mathcal{L}_n^{(1)}$ are derived in accordance with the block structure of the inverse of the matrix $\mathcal{L}_n^{(1)}$. Additional combinatorial identities identities involving the Lucas numbers are derived by observing the particular case $x = 1/2$, $a = 2$, $b = 1$ we derived.

2. DEFINITIONS AND MOTIVATION

For the sake of completeness, we restate basic facts about the special functions needed in the present paper. The Euler gamma function is denoted by $\Gamma(n)$,

$$(a)_n = a(a+1)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$$

is well-known Pochhammer function,

$${}_pF_q(a_1, \dots, a_p; b_1, \dots b_q; \lambda) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \cdot \frac{\lambda^k}{k!}$$

is a generalized hypergeometric function, and

$${}_p\tilde{F}_q[\{a_1, \dots, a_p\}, \{b_1, \dots, b_q\}, z] = {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) \,/\, (\Gamma(b_1) \dots \Gamma(b_q))$$

defines the regularized generalized hypergeometric function.

Various types of lower triangular Toeplitz matrices which include various types of combinatorial numbers were investigated in [1, 2, 4, 6, 24, 25]. The generalized Pascal matrix of the first kind $\mathcal{P}_n[x] = [p_{i,j}[x]]$, $i, j = 1, \dots, n$ is defined in [4]:

$$p_{i,j}[x] = \begin{cases} x^{i-j}\binom{i-1}{j-1} & i \geq j, \\ 0 & i < j. \end{cases} \tag{2.1}$$

In the case $x = 1$, the generalized Pascal matrix of the first kind reduces to frequently used Pascal matrix $\mathcal{P}_n = [p_{i,j}]$, $i, j = 1, \dots, n$, which is defined in [3, 4].

The $n \times n$ Fibonacci matrix $\mathcal{F}_n = [f_{i,j}]$ $(i, j = 1, \dots, n)$ is defined in [11], arranging the Fibonacci numbers on the main diagonal and below :

$$f_{i,j} = \begin{cases} F_{i-j+1}, & i-j+1 \geq 0, \\ 0, & i-j+1 < 0 \end{cases}. \tag{2.2}$$

The inverse and Cholesky factorization of the Fibonacci matrix are given in [11]. A first kind as well as the second kind factorization of the Pascal matrix in terms of the Fibonacci matrix are studied in [12]. Very helpful consequences of these matrix relations are various combinatorial identities involving the binomial coefficients and the Fibonacci numbers.

As an analogy of the Fibonacci matrix, the $n \times n$ Lucas matrix $\mathcal{L}_n = [l_{i,j}]$ $(i, j = 1, \dots, n)$ is defined in [27]:

$$l_{i,j} = \begin{cases} L_{i-j+1}, & i-j \geq 0, \\ 0, & i-j < 0. \end{cases} \tag{2.3}$$

In the particular case $a = 2$, $b = 1$ the generalized Fibonacci matrix from [19] reduces to a generalization of the Lucas matrix.

Definition 2.1 *The Lucas matrix of type s and of the order n, denoted by* $\mathcal{L}_n^{(s)} = \left[l_{i,j}^{(s)}\right]$, *i, j = 1, ..., n, is given by*

$$l_{i,j}^{(s)} = \begin{cases} L_{i-j+1}, & i-j+s \geq 0 \\ 0, & i-j+s < 0 \end{cases} \quad i, j = 1, \ldots, n. \tag{2.4}$$

The only two cases that generate regular matrices of this type are $s = 0$ and $s = 1$ (for the proof see [19]). In this paper we consider the regular matrix $\mathcal{L}_n^{(1)}$, corresponding to the choice $s = 1$ in (2.4). The case $s = 0$ is investigated in [27]. In all other cases the matrix $\mathcal{L}_n^{(s)}$ is singular. These cases are not considered in [27].

In the study [22], some new properties of Lucas numbers with binomial coefficients were obtained to write Lucas sequences in a new direct way. Sums of squares of odd and even terms of the Lucas sequence and alternating sums of their products were investigated in [5]. In the paper [7], some known fibonacci and Lucas sums were derived by using proofs based on the usage of two appropriate matrices. A few representations of Lucas numbers in terms of the generalized hypergeometric functions were derived in [16]. Several combinatorial identities involving binomial coefficients, powers of two and the Lucas numbers are known. For example, we restate identities from [8]:

$$\sum_{k=0}^{n} \binom{n}{k} 2^k L_k = L_{3n}, \quad \sum_{k=0}^{n} \binom{n}{k} L_k = L_{2n}.$$

In the present paper we derive interesting representations for sums of the form $\Sigma 2^{k-1} L_k$ and $\Sigma k \cdot 2^{k-1} L_k$. Further, we derive various more general combinatorial identities involving Lucas numbers, binomial co-effcients and some special functions. These identities are derived continuing the general principles from [12, 20, 18, 26, 27] and using explicit representation of the inverse of the matrix $\mathcal{L}_n^{(1)}$ and the first kind factorization of the Pascal matrix.

3. SOME IDENTITIES OF SUMS INVOLVING LUCAS NUMBERS

The following combinatorial identity gives the exact value of the sum $\Sigma(n) = \sum_{k=0}^{n} 2^{k-1} L_k$.

Lemma 3.1 *The following recurrence relation is valid for arbitrary integer $n \geq 0$.*

$$\Sigma(n) = \sum_{k=0}^{n} 2^{k-1} L_k = \frac{2^n}{5}(2L_{n+2} - L_{n+1}) \tag{3.1}$$

$$= 2^n F_{n+1}. \tag{3.2}$$

Proof. The identity (3.1) can be verified by the principle of the mathematical induction. The inductive step follows from

$$\sum_{k=0}^{n+1} 2^{k-1} L_k = 2^n L_{n+1} + \frac{2^n}{5}(2L_{n+2} - L_{n+1})$$

$$= \frac{2^n}{5}(2L_{n+2} + 4L_{n+1}) = \frac{2^{n+1}}{5}(2L_{n+3} - L_{n+2}).$$

Finally, using

$$2L_{n+2} - L_{n+1} = L_{n+2} - L_n = 5F_{n+1}$$

the identity (3.2) is verified.

Corollary 3.1. *For arbitrary integer $n \geq 0$, the next identities hold:*

$$\Sigma(n) = \frac{1}{40}\left(1-\sqrt{5}\right)^{-n-2}\left(-\left(\sqrt{5}-5\right)(-4)^{n+2} + \left(5+\sqrt{5}\right)\left(6-2\sqrt{5}\right)^{n+2}\right) \tag{3.3}$$

$$= \frac{\sqrt{5}}{10}\left(\left(1+\sqrt{5}\right)^{n+1} - \left(1-\sqrt{5}\right)^{n+1}\right). \tag{3.4}$$

Proof. The proof can be completed applying the analog of Binet's Fibonacci number formula for Lucas numbers:

$$L_k = \left(\frac{1-\sqrt{5}}{2}\right)^k + \left(\frac{1+\sqrt{5}}{2}\right)^k, \; k \geq 0. \tag{3.5}$$

to (3.1) we get.

Corollary 3.2 *The following representation of Lucas numbers is valid for arbitrary integer* $n \geq 0$:

$$F_{n+1} = \frac{\sum_{k=0}^{n} {}_2F_1\left(-\frac{1}{2}k, -\frac{1}{2}k+\frac{1}{2}; \frac{1}{2}; 5\right)}{2^n} \tag{3.6}$$

Proof. The proof can be derived applying the next representation of the Lucas numbers from [16]

$$L_k = \left(\frac{1}{2}\right)^{k-1} {}_2F_1\left(-\frac{1}{2}k, -\frac{1}{2}k+\frac{1}{2}; \frac{1}{2}; 5\right)$$

to (3.2).

Proposition 3.1 *(a) The sum* $2\Sigma(n)$ *ends with* mod$(2(n+1), 10)$.

(b) The sum $\Sigma(n)$ *ends with* mod$(6n-4, 10)$.

Proof. (a) Since each term $2^k L_k$ ends with 2, for each integer k (see [8]), the expression $2\Sigma(n)$ contains $n+1$ terms obeying this property.

In the following lemma we introduce an additional combinatorial identity involving the Lucas numbers.

Lemma 3.2. *Consider integers* i, j *and* s, p *which satisfy* $p \geq 3$, $s \leq p \leq i+1$. *The partial convolution defined by*

$$I_1(s, p, i, j) = \sum_{k=s}^{p} 2^{j-k} L_{i-k+1} = 2^{j-1} \sum_{k=s-1}^{p-1} 2^{-k} L_{i-k} \tag{3.7}$$

satisfies the following equality :

$$I_1(s, p, i, j) = \begin{cases} 2^{j-i}\left(\Sigma(i-s+1) - \Sigma(i-p)\right), & p \leq i, \\ 2^{j-i}\Sigma(i-s+1), & p = i+1. \end{cases} \tag{3.8}$$

Proof. The first case of the proof follows from

$$\sum_{k=s}^{p} 2^{j-k} L_{i-k+1} = 2^{j-i} \sum_{k=s}^{p} 2^{i-k} L_{i-k+1} = 2^{j-i} \sum_{j=i-p+1}^{i-s+1} 2^{j-1} L_j$$

$$= 2^{j-i}\left(\sum_{j=0}^{i-s+1} 2^{j-1} L_j - \sum_{j=0}^{i-p} 2^{j-1} L_j \right)$$

and (3.1). Since

$$I_1(s, i+1, i, j) = I_1(s, i, i, j) + 1 = \Sigma(i-s+1) - \Sigma(i-i) + 1 = \Sigma(i-s+1),$$

the second case is also verified.

Now, we find explicit representation of the sum

$$\sum_{k=0}^{n} k \cdot 2^{k-1} L_k .$$

Theorem 3.1 *The following indentity is valid for arbitrary integer* $n \geq 1$:

$$\Sigma_1(n) = \sum_{k=0}^{n} k \cdot 2^{k-1} L_k = \frac{1 + 2^n((n-1)L_{n+2} + (n+1)L_n)}{5} \tag{3.9}$$

$$= \frac{1 + 2^n((3n+5)L_n + (n+1)L_{n-1})}{5} - 2 \cdot \Sigma(n) . \tag{3.10}$$

Proof. The identity (3.9) is verified by the mathematical induction. The base of the induction is easy for the verification and the inductive step follows from

$$\Sigma_1(n) = (n+1)2^n L_{n+1} + \frac{1 + 2^n((n-1)L_{n+2} + (n+1)L_n)}{5}$$

$$= \frac{1 + 2^n((n-1)L_{n+2} + (5n+5)L_{n+1} + (n+1)L_n)}{5}$$

$$= \frac{1 + 2^n(2nL_{n+3} - (n+1)L_{n+2} + (3n+5)L_{n+1} + (n+1)L_n)}{5}$$

$$= \frac{1+2^n(2nL_{n+3}+(2n+4)L_{n+1})}{5}.$$

Identity (3.10) is a consequence of (3.9).

4. IDENTITIES BASED ON THE USAGE OF GENERALIZED PASCAL MATRIX

In this section we investigate correlations between the matrix $\mathcal{L}_n^{(1)}$ and the generalized Pascal matrices. Several new combinatorial identities are derived as implications.

Theorem 4.1. *The inverse* $\left(\mathcal{L}_n^{(1)}\right)^{-1} = [l'_{i,j}]$ $(i, j = 1, ..., n)$ *of the Lucas matrix* $\mathcal{L}_n^{(1)}$ *is defined by*

$$l'_{i,j} = \begin{cases} 3, & i>2,\ j\le n-2,\ i=j+1; \\ 5\cdot 2^{j-i}, & i>2,\ j\le n-2,\ i\le j; \\ (-1)^i\cdot 2^{j-i}, & i\le 2,\ j\le n-2,\ (i,j)\ne(2,1); \\ (-1)^{n-j+1}\cdot 2^{j-i}, & i>2,\ j>n-2,\ (i,j)\ne(n,\ n-1); \\ (-1)^{n-j+i-1}\cdot \frac{2^{j-i}}{5}, & i\le 2,\ j>n-2; \\ 0, & i>j+2; \\ 1, & otherwise. \end{cases} \tag{4.1}$$

Proof. To simplify notation, let us denote $c_{i,j} = \sum_{k=1}^{n} l_{i,k}l'_{k,j}$. First, we want to prove $c_{i,i} = 1$, $i = 1, 2, ..., n$. Three different possibilities are feasible.

(C_1) : It is easy to prove that $c_{1,1} = 1$. In the case $i = 2$ immediately follows

$$c_{2,2} = -2L_2 + L_1 + 3L_0 = 1.$$

(C_2) : In the cae $2 < i \le n-2$, the value $c_{i,i}$ is equal to

$$c_{i,i} = \sum_{k=1}^{n} l_{i,k}l'_{k,i} = l_{i,1}l'_{1,i} + l_{i,2}l'_{2,i} + \sum_{k=3}^{i} l_{i,k}l'_{k,i} + l_{i,i+1}l'_{i+1,i}$$

$$= -2^{i-1}L_i + 2^{i-2}L_{i-1} + \sum_{k=3}^{i} 5\cdot 2^{i-k}L_{i-k+1} + 3L_0.$$

The result of Lemma 3.2 implies

$$\sum_{k=3}^{i} 5\cdot 2^{i-k} L_{i-k+1} = 5I_1(3,\ i,\ i,\ i) = 5(\Sigma(i-2)-1)$$

and later, taking into account (3.1), one can verify

$$c_{ij} = -5\Sigma(i-2) + 5(\Sigma(i-2)-1) + 3L_0 = 1.$$

(C_3) : Finally, in the case $i = n$ (proof for $i = n-1$ is very similar), it follows that

$$c_{n,n} = \sum_{k=1}^{n} l_{n,k} l'_{k,n} = l_{n,1} l'_{1,n} + l_{n,2} l'_{2,n} + \sum_{k=3}^{n} l_{n,k} l'_{k,n}$$

$$= \frac{2^{n-1}}{5} L_n - \frac{2^{n-2}}{5} L_{n-1} - \sum_{k=3}^{n} 2^{n-k} L_{n-k+1}.$$

Applying the results of Lemma 3.1 and Lemma 3.2, this part of the proof can be simply completed.

Now, we want to prove that $c_{i,j} = 0$, for $i \neq j$. Similarly as in the first part of the proof, several different cases can be distinguished.

(D_1) : It is easy to see that $c_{i,1} = 0$ in the case $i > 1$.

(D_2) : In the case $1 < j < n-1$, $i > j$ the following holds:

$$c_{i,j} = \sum_{k=1}^{n} l_{i,k} l'_{k,j} = l_{i,1} l'_{1,j} + l_{i,2} l'_{2,j} + \sum_{k=3}^{j} l_{i,k} l'_{k,j} + l_{i,j+1} l'_{j+1,j} + l_{i,j+2} l'_{j+2,j}$$

$$= -2^{j-1} L_i + 2^{j-2} L_{i-1} + 5\sum_{k=3}^{j} L_{i-k+1} 2^{j-k} + 3L_{i-j} + L_{i-j-1}.$$

Applying the result of Lemma 3.2 on the sum

$$\sum_{k=3}^{j} 2^{j-k} L_{i-k=1} = I_1(3,\ j,\ i,\ j),$$

one can verify

$$c_{i,j} = -2^{j-i}\Sigma(i-2) + 5\cdot 2^{j-i}(\Sigma(i-2) - \Sigma(i-j)) + 3L_{i-j} + L_{i-j-1}$$

$$= -5\cdot 2^{i-i}\cdot\frac{2^{i-j}}{5}(2L_{i-j+2} - L_{i-j+1}) + 2L_{i-j} + L_{i-j+1} = 0.$$

(D_3) : In the case $1 < j < n-1$, $i < j$, values $c_{i,j}$ are defined as

$$c_{i,j} = l_{i,1}l'_{1,j} + l_{i,2}l'_{2,j} + \sum_{k=3}^{i+1} l_{i,k}l'_{k,j} = -2^{j-1}L_i + 2^{j-2}L_{i-1} + 5\sum_{k=3}^{i+1} L_{i-k+1}2^{j-k}.$$

An application of the second case of (3.8) in conjunction with the results of Lemma 3.2, further implies :

$$c_{i,j} = -5\cdot 2^{j-i}(\Sigma(i-2) - \Sigma(i-2)) = 0.$$

(D_4) : Finally, in the case $j = n$ (proof for $j = n-1$ can be accomplished similarly), values $c_{i,n}$ are equal to

$$c_{i,n} = l_{i,1}l'_{1,n} + l_{i,2}l'_{2,n} + \sum_{k=3}^{i+1} l_{i,k}l'_{k,n} = \frac{1}{5}\left(2^{n-1}L_i - 2^{n-2}L_{i-1}\right) - \sum_{k=3}^{i+1} 2^{n-k}L_{i-k+1}.$$

Using the result from Lemma 3.2 together with the second case of (3.8) we have

$$5\sum_{k=3}^{i+1} 2^{n-k}L_{i-k+1} = I_1(3, i+1, i, n) = 2^{n-i}\Sigma(i-2),$$

which implies $c_{i,n} = 0$.

Hence, we prove $\mathcal{L}_n^{(1)}\left(\mathcal{L}_n^{(1)}\right)^{-1} = I_n$, where I_n is the $n \times n$ identity matrix. In a similar way one can verify $\left(\mathcal{L}_n^{(1)}\right)^{-1}\mathcal{L}_n^{(1)} = I_n$ and the proof is completed.

According to Lemma 4.1, it is observable that $\left(\mathcal{L}_n^{(1)}\right)^{-1}$ possesses the following block matrix form :

$$\left(\mathcal{L}_n^{(1)}\right)^{-1} = \left[\begin{array}{c|c} \left(\mathcal{L}_n^{(1)}\right)^{-1}{}_{2|,(n-2)|} & \left(\mathcal{L}_n^{(1)}\right)^{-1}{}_{2|,|2} \\ \hline \left(\mathcal{L}_n^{(1)}\right)^{-1}{}_{|(n-2),(n-2)|} & \left(\mathcal{L}_n^{(1)}\right)^{-1}{}_{|(n-2),|2} \end{array}\right], \tag{4.2}$$

where the notations $k|$ (resp. $|k$) in the first indices denote the first (resp. last) k rows and $k|$ (resp. $|k$) in the second indices denote the first (resp. last) k columns.

Example 4.1 *The* 6×6 *inverse Lucas matrix of type* 1 *is equal to*

$$\left(\mathcal{L}_6^{(1)}\right)^{-1} = \left[\begin{array}{cccc|cc} -1 & -2 & -4 & -8 & -\frac{16}{5} & -\frac{32}{5} \\ 1 & 1 & 2 & 4 & \frac{8}{5} & -\frac{16}{5} \\ \hline 1 & 3 & 5 & 10 & 4 & -8 \\ 0 & 1 & 3 & 5 & 2 & -4 \\ 0 & 0 & 1 & 3 & 1 & -2 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right].$$

It is interesting to point out that the blocks of the Lucas inverse matrix of type 1 are almost constant along diagonals, which means that the matrix $\left(\mathcal{L}_n^{(1)}\right)^{-1}$ is close to block Toeplitz matrix.

Although (4.1) looks robust, it is not difficult to see that three blocks of the matrix $\left(\mathcal{L}_n^{(1)}\right)^{-1}$ are symmetric in relation to anti-diagonal and are almost Toeplitz (differs only in signs).

Remark 4.1 *To store the matrix* $\left(\mathcal{L}_n^{(1)}\right)^{-1}$ *we only need to register in the memory: first row, element in position* $(2, n-1)$ *and first row of the* $(n-2) \times (n-2)$ *toeplitz matrix which is in the left-lower block. This means that we have to store only the vector with* $(2n-1)$ *elements, which is the same size for storing arbitrary Toeplitz matrix.*

A factorization of the Pascal matrix given in terms of the Lucas matrix of type 1 is presented in the following lemma.

Lemma 4.1 *For arbitrary integer* $n > 4$, *the generalized Pascal matrix and the Lucas matrix of type* 1 *satisfy*

$$\mathcal{P}_n[1/2] = \mathcal{H}_n^{(1)}[1/2]\mathcal{L}_n^{(1)},$$

where the matrix

$$\mathcal{H}_n^{(1)}[1/2] = \left[h_{i,j}^{(1)}\left(\tfrac{1}{2}\right)\right]$$

is defined by

$$h_{i,j}^{(1)}\left(\frac{1}{2}\right) = \begin{cases} 2^{j-i}\left(5\cdot 2^{i-1} - 4i - 2 + 6\binom{i-1}{j} + 4\binom{i-1}{j+1} - 5\binom{i-1}{j}{}_2F_1(1, j-i+1; j+1; -1)\right) & 1 < j \le n-2; \\ 2^{j-1}(-1)^{n-j}\left(\frac{1+2i}{5}2^{-i+2} - 1\right), & j > n-2, (i,j) \ne (n, n-1); \\ 2^{n-2} - \frac{3+n}{10}, & (i,j) = (n, n-1); \\ 2^{1-i}(1 + 2i(i-2)), & j = 1. \end{cases} \tag{43}$$

Proof. It suffices to verify the matrix equality $\mathcal{H}_n^{(1)}[1/2] = \mathcal{P}_n[1/2]\left(\mathcal{L}_n^{(1)}\right)^{-1}$. Let us denote by $c_{ij} = \sum_{k=1}^{n} p_{i,k} l'_{k,j}$ and observe analogous cases as in (4.3).

$$\begin{aligned} c_{ij} &= \sum_{k=1}^{n} p_{i,k} l'_{k,j} = p_{i,1} l'_{1,j} + p_{i,2} l'_{2,j} + \sum_{k=3}^{j} p_{i,k} l'_{k,j} + p_{i,j+1} l'_{j+1,j} + p_{i,j+2} l'_{j+2,j} \\ &= -2^{j-i} + (i-1)2^{j-i} + 5\sum_{k=3}^{j}\binom{i-1}{k-1}2^{j-i} + 3\binom{i-1}{j}2^{j-i+1} + \binom{i-1}{j+1}2^{j-i+2} \\ &= 2^{j-i}\left[i - 2 + 5\sum_{k=3}^{j}\binom{i-1}{k-1} + 6\binom{i-1}{j} + 4\binom{i-1}{j+1}\right] \end{aligned}$$

After the application of

$$\begin{aligned} \sum_{k=3}^{j}\binom{i-1}{k-1} &= 2^{i-1} - i - \frac{\Gamma(i)\,{}_2\tilde{F}_1[\{1, -i+j+1\}\cdot\{j+1\}, -1]}{\Gamma(i-j)} \\ &= 2^{i-1} - i - \binom{i-1}{j}\,{}_2F_1(1, j-i+1, j+1, -1), \end{aligned}$$

the first case can be verified.

For $j > n-2$, $(i, j) \ne (n, n-1)$, we have

$$c_{i,j} = \sum_{k=1}^{n} p_{i,k} l'_{k,j} = p_{i,1} l'_{1,j} + p_{i,2} l'_{2,j} + \sum_{k=3}^{i} p_{i,k} l'_{k,j}$$

$$= \left(\frac{1}{2}\right)^{i-1} \frac{(-1)^{n-j} 2^{j-1}}{5} + (i-1)\left(\frac{1}{2}\right)^{i-2} \frac{(-1)^{n-j+1} 2^{j-2}}{5} + (-1)^{n-j+1} \sum_{k=3}^{i} \binom{i-1}{k-1} \left(\frac{1}{2}\right)^{i-k} 2^{j-k}$$

$$= 2^{j-i}(-1)^{n-j}\left[\frac{1}{5} - \frac{i-1}{5} - \sum_{k=3}^{i} \binom{i-1}{k-1}\right].$$

The approval follows straight having in mind the identity

$$\sum_{k=3}^{i} \binom{i-1}{k-1} = 2^{i-1} - i.$$

In the case $(i, j) = (n, n-1)$, we have

$$c_{n,\, n-1} = \sum_{k=1}^{n} p_{n,k} l'_{k,n-1} = p_{n,1} l'_{1,n-1} + p_{n,2} l'_{2,n-1} + \sum_{k=3}^{n-1} p_{n,k} l'_{k,n-1} + 1$$

$$= -\left(\frac{1}{2}\right)^{n-1} \frac{2^{n-2}}{5} + (n-1)\left(\frac{1}{2}\right)^{n-2} \frac{2^{n-3}}{5} + \sum_{k=3}^{n-1} \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-k} 2^{n-k-1} + 1$$

$$= -\frac{2}{5} + \frac{2}{5}(n-1) + \frac{1}{2} \sum_{k=3}^{n-1} \binom{n-1}{k-1} + 1.$$

After applying

$$\sum_{k=3}^{n-1} \binom{n-1}{k-1} = 2^{n-1} - n - 1, \; n > 4,$$

it is possible to obtain

$$c_{i,j} = 2^{n-2} - \frac{3+n}{10}.$$

It is obvious that for $j = 1$ the following is valid

$$c_{i,1} = 2^{3-i}\left(-\frac{1}{4} + \frac{1}{2}\binom{i-1}{1} + \binom{i-1}{2}\right),$$

which immediately produces the last case in (4.3). We get that $c_{i,j} = h^{(1)}_{i,j}(1/2)$ for $i, j = 1, \ldots, n$ and the proof is completed.

The next identity follows from Lemma 4.1.

Theorem 4.2 *The following recurrence relation is valid for arbitrary integers* $n \geq 2$, $1 \leq i \leq n-1$:

$$2i(i-2) + \sum_{j=2}^{n-2} 2^{j-1}\left(5\cdot 2^{i-1} - 4i - 2 + 6\binom{i-1}{j} + 4\binom{i-1}{j+1} - 5\binom{i-1}{j}\,{}_2F_1(1, j-i+1; j+1; -1)\right)L_j$$

$$+ \sum_{j=n-1}^{n} 2^{j-1}(-1)^{n-j}\left(\frac{2-i}{5} - \sum_{k=3}^{i}\binom{i-1}{k-1}\right)L_j = 0. \tag{4.4}$$

Proof. The following notation is useful:

$$p_{i,j} = p_{i,j}[1/2],\ h_{i,j} = h^{(1)}_{i,j}(1/2).$$

The next identity could be derived from Lemma 4.1, taking into account (4.3) :

$$\left(\frac{1}{2}\right)^{i-1} = p_{i,1} = \sum_{j=1}^{n} h_{i,j}l_{j,1} = h_{i,1}l_{1,1} + \sum_{j=2}^{n-2} h_{i,j}l_{j,1} + \sum_{j=n-1}^{n} h_{i,j}l_{j,1}$$

$$= 2^{1-i}(1 + 2i(i-2)) +$$

$$+ \sum_{j=2}^{n-2} 2^{j-i}\left(5\cdot 2^{i-1} - 4i - 2 + 6\binom{i-1}{j} + 4\binom{i-1}{j+1} - 5\binom{i-1}{j}\,{}_2F_1(1, j-i+1; j+1; -1)\right)L_j$$

$$+ \sum_{j=n-1}^{n} 2^{j-i}(-1)^{n-j}\left(\frac{2-i}{5} - \sum_{k=3}^{n-1}\binom{i-1}{k-1}\right)L_j.$$

In the case $1 \leq i \leq n-1$, (4.4) immediately follows.

Remark 4.2 (i) *In the case* $i = 1$, *the identity* (4.4) *reduces to* (3.1).

(ii) *In the case* $i = 2$, *the identity* (4.4) *becomes the trivial identity* $0 = 0$.

5. CONCLUSION

Several combinatorial identities involving Lucas numbers, Fibonacci numbers and exponents of two are derived by the mathematical induction. The starting point of our investigation is the notion of the Lucas matrix $\mathcal{L}_n^{(s)}$ of type s, which is constant along the diagonals and filled with the generalized Fibonacci numbers. This matrix introduced in [19]. The regular case $s = 0$ includes the results obtained in [19]. In the present paper, the case $s = 1$ is investigated. The inverse of the Lucas matrix $\mathcal{L}_n^{(1)}$ is generated using identities previously derived by the induction.

The principles used in the papers [12, 19, 18, 26] are applied in sections 3 and 4, following the specifies of these particular kinds of Toeplitz matrices. Using the identity for the sum involving generalized Fibonacci numbers, derived in Lemma 3.2, the explicit representation for the inverse matrix of $\mathcal{L}_n^{(1)}$ is given in Lemma 4.1. A first kind factorization of the generalized Pascal matrix in terms of the Lucas matrix $\mathcal{L}_n^{(1)}$ is derived in Theorem 4.1. Based upon these matrix correlations, several interesting additional combinatorial identities involving the Lucas numbers and the binomial coefficients are derived, observing the special case $x = 1/2$, $a = 2$, $b = 1$.

Remaining singular cases of these matrices ($s \neq 0, 1$), could be the interest of our future research.

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