Jour. Pure Math., Vol. 24, 2007, pp. 31-38

WEAKLY CONTRA-β-CONTINUOUS FUNCTIONS AND STRONGLY Sβ-CLOSED SETS

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ABSTRACT : A new form of contra-β-continuity, called weak contra-β-continuity, is introduced. We show that this class of function is weaker than most forms of β-continuity and contra-continuity, but that the class still has interesting properties. The notion of a strongly Sβ-closed space is defined and conditions are established under which the weakly contra-β-continuous image of a strongly Sβ-closed space is compact. Various properties of strongly Sβ-closed spaces are investigated. For example, we show that every strongly Sβ-closed is the β-closure of a finite set.

Key words and phrases : contra-continuity, β-continuity, contra-β-continuity, weakly contra-β-continuity, nearly compact spaces, strongly Sβ-closed spaces.

2000 Mathematics Subject Classification: 54C10, 54D10.

1. INTRODUCTION

The notion of a β-continuous function was introduced by Abd El-Monsef, et al. [1] in 1983. Over the intervening years many variations of β-continuity have been developed. For example, Noiri and Popa [16] developed the concept of weak β-continuity in 2000 and Noiri [15] introduced slightly β-continuous functions in 2001. Also much work has been done on the notion of contra-continuity and its variations. Contra-continuity was developed by Dontchev [8] in 1996. Dontchev investigated relationships between contra-continuity and coverings of spaces by closed sets. Specifically he defined the notion of a strongly S-closed space and developed conditions under which contra-continuous images of S-closed and strongly S-closed spaces are compact. Caldas and Jafari [6] introduced contra-β-continuity in 2001 and Jafari and Noiri [10] developed contra-precontinuity in 2002. Recently Baker [3, 4, 5] introduced contra-almost β-continuity, weak contra-continuity and weak contra-precontinuity. The purpose of this note is to introduce a form of contra-continuity, which we call weak contra-β-continuity, and to investigate relationships between weak contra-beta-continuity and covers of spaces by β-closed sets. We show weak contra-β-continuity is weaker than β-continuity, contra-β-continuity and weak contra-precontinuity but stronger than slight β-continuity. The concept of a strongly Sβ-closed space is developed and conditions are established under which weakly contra-β-continuous images of strongly Sβ-closed spaces and nearly compact spaces are compact.

2. PRELIMINARIES

The symbols X and Y represent topological spaces with no separation properties assumed unless explicitly stated. All sets are considered to be subsets of topological spaces. The closure and interior of a set A are signified by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A set A is regular open (respectively, β-open [1], α-open [14], semi-open [11], preopen [13]) provided that $A = \mathrm{Int}(\mathrm{Cl}(A))$ (respectively $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$, $A \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$, $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$, $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$). A set is regular closed (respectively, β-closed, α-closed, semi-closed, preclosed) if its complement is regular open (respectively, β-open, α-open, semi-open, preopen). The β-closure of a set A, denoted by $\beta\mathrm{Cl}(A)$, is the intersection of all β-closed sets containing A. The preclosure of a set A, denoted by $\mathrm{pCl}(A)$, is defined analogously using preclosed sets.

Definition 1. *A function $f : X \to Y$ is said to be contra-continuous* [8] *if $f^{-1}(V)$ is closed for every open subset V of Y.*

Definition 2. *A function $f : X \to Y$ is said to be β-continuous* [1] *if $f^{-1}(V)$ is β-open for every open subset V of Y.*

Definition 3. *A function $f : X \to Y$ is said to be contra-β-continuous* [6] *if $f^{-1}(V)$ is β-closed for every open subset V of Y.*

Definition 4. *A function $f : X \to Y$ is said to be weakly contra-continuous* [4] *provided that, whenever $A \subseteq V \subseteq Y$, A is closed in Y, and V is open Y, then $Cl(f^{-1}(A)) \subseteq f^{-1}(V)$.*

Definition 5. *A function $f : X \to Y$ is said to be weakly contra-precontinuous* [5] *provided that, whenever $A \subseteq V \subseteq Y$, A is closed in Y, and V is open in Y, then $pCl(f^{-1}(A)) \subseteq f^{-1}(V)$.*

Definition 6. *A function $f : X \to Y$ is said to be slightly β-continuous* [17] *provided that, for every $x \in X$ and every clopen subset V of Y containing $f(x)$, there exists a β-open subset U of X such that $x \in U$ and $f(U) \subseteq V$.*

3. WEAKLY CONTRA-β-CONTINUOUS FUNCTIONS

We define a function $f : X \to Y$ to be weakly contra-β-continuous provided that, if $A \subseteq V \subseteq Y$, with A closed in Y and V open in Y, then $\beta\mathrm{Cl}(f^{-1}(A)) \subseteq f^{-1}(V)$. Since $\beta\mathrm{Cl}(f^{-1}(A)) = f^{-1}(A) \cup \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(f^{-1}(A))))$, we have the following characterization of weak contra-β-continuity.

Theorem 3.1. *A function $f : X \to Y$ is weakly contra-β-continuous if and only if, whenever A is a closed subset of Y, V is an open subset of Y, and $A \subseteq V$, then $Int(Cl(Int(f^{-1}(A)))) \subseteq f^{-1}(V)$.*

Theorem 3.2. *If $f : X \to Y$ is β-continuous, then f is weakly contra-β-continuous.*

Proof. Assume $f : X \to Y$ is β-continuous and let $A \subseteq V \subseteq Y$, with A closed in Y and V open in Y. Then, since $f^{-1}(A)$ is β-closed, we have $\beta Cl(f^{-1}(A)) = f^{-1}(A) \subseteq f^{-1}(V)$, which proves that f is weakly contra-β-continuous.

Theorem 3.3. *If $f : X \to Y$ is contra-β-continuous, then f is weakly contra-β-continuous.*

Proof. Assume $f : X \to Y$ is contra-β-continuous and let $A \subseteq V \subseteq Y$, with A closed in Y and V open in Y. Then, since f is contra-β-continuous, $f^{-1}(V)$ is β-closed. Therefore $\beta Cl(f^{-1}(A)) \subseteq \beta Cl(f^{-1}(V)) = f^{-1}(V)$ and hence f is weakly contra-β-continuous.

Since contra-β-continuity and β-continuiy are independent [6], it follows that weak contra-β-continuity does not imply either β-continuity or contra-β-continuity.

The following result is a partial converse of Theorem 3.3.

Definition 7. *A function $f : X \to Y$ is said to be subcontra-β-continuous provided there exists an open base $\mathcal{B}$ for the topology on Y such that $f^{-1}(V)$ is β-closed for every $V \in \mathcal{B}$.*

Recall that a space is called zero dimensional provided it has a clopen base.

Theorem 3.4. *If $f : X \to Y$ is weakly contra-β-continuous and Y is zero dimensional, then f is subcontra-β-continuous.*

Proof. Let $\mathcal{B}$ be a clopen base for Y and let $B \in \mathcal{B}$. Since f is weakly contra-β-continuous, we see that $\beta Cl(f^{-1}(B)) \subseteq f^{-1}(B)$ and hence that $f^{-1}(B)$ is β-closed.

Since $\beta Cl(A) \subseteq pCl \subseteq Cl(A)$ for every set A, weak contra-continuity ⇒ weak contra-precontinuity ⇒ weak contra-β-continuity. From [5] weak contra-precontinuity does not imply weak contra-continuity and the following example shows that weak contra-β-continuity does not imply weak contra-precontinuity.

Example 3.5. *Assume the space $X = \{a, b, c\}$ has the topologies $\sigma = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}$ and $\tau = \{X, \emptyset, \{b\}, \{a, b\}, \{b, c\}\}$. Then the identity mapping $f : (X, \sigma) \to (X, \tau)$ is weakly contra-β-continuous but not weakly contra-precontinuous.*

Theorem 3.6. *If $f : X \to Y$ is weakly contra-β-continuous, then f is slightly β-continuous.*

Proof. Let V be a clopen subset of Y. Then we have $V \subseteq V \subseteq Y$ and, since f is weakly contra-β-continuous, $\beta Cl(f^{-1}(V)) \subseteq f^{-1}(V)$. Therefore $f^{-1}(V)$ is β-closed, which proves that f is slightly β-continuous.

The following example shows that the implication in Theorem 3.6 can not be reversed.

Example 3.7. [5] *Let X denote the real numbers, let $\sigma = \{X, \emptyset, \{0\}\}$, and let τ be the usual topology on X. Since (X, τ) is connected, the identity mapping $f : (X, \sigma) \to (X, \tau)$ is slightly β-continuous. However, since $\{0\} \subseteq (-1, 1) \subseteq (X, \tau)$ with $\{0\}$ closed and $(-1, 1)$ open, but $\beta Cl(f^{-1}(\{0\})) \not\subseteq f^{-1}(-1, 1)$, f fails to be weakly contra-β-continuous.*

We have the following implications, none of which are reversible:

$$\begin{array}{c} \text{Weak contra-continuity} \\ \Downarrow \\ \text{weak contra-precontinuity} \\ \Downarrow \\ \beta\text{-continuous} \Rightarrow \text{weak contra-}\beta\text{-continuous} \Rightarrow \text{slight } \beta\text{-continuity} \\ \Uparrow \\ \text{contra-}\beta\text{-continuity} \end{array}$$

We now investigate relationships between weak contra-β-continuity and various forms of β-continuity and a type of generalized β-continuity.

Definition 8. *A function $f : X \to Y$ is said to be contra-almost β-continuous* [3] *provided that $f^{-1}(V)$ is β-closed for every regular open subset V of Y.*

Recall that a space is extremally disconnected (briefly an ED space) provided that closures of open sets are open.

Theorem 3.8. *If $f : X \to Y$ is weakly contra-β-continuous and Y is an ED space, then f is contra-almost β-continuous.*

Proof. Let V be a regular open subset of Y. Since Y is an ED space, V is also closed. Then, because f is weakly contra-β-continuous, $\beta Cl(f^{-1}(V)) \subseteq f^{-1}(V)$. Therefore $f^{-1}(V)$ is β-closed, which proves that f is contra-almost β-continuous.

Definition 9. *A function $f : X \to Y$ is said to be weakly β-continuous* [17] *provided that, for every $x \in X$ and every open subset V of Y containing $f(x)$, there exists a β-open subset U of X with $x \in U$ such that $f(U) \subseteq Cl(V)$.*

Since contra-almost β-continuity implies weak β-continuity [3] we have the following corollary.

Corollary 3.9. *If $f : X \to Y$ is weakly contra-β-continuous and Y is an ED space, then f is weakly β-continuous.*

Definition 10. *A subset A of a space X is said to be generalized β-closed (briefly gβ-closed) (also called generalized semi-preclosed)* [7] *provided that $\beta Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.*

Definition 11. *A function $f : X \to Y$ is called generalized β-continuous (briefly gβ-continuous) (also called gsp-continuous)* [7] *if $f^{-1}(V)$ is gβ-closed in X for every closed subset V of Y.*

Definition 12. *A function $f : X \to Y$ is said to be approximately β-continuous (briefly aβ-continuous) provided that $\beta Cl(A) \subseteq f^{-1}(V)$ whenever V is open in Y, A is gβ-closed in X, and $A \subseteq f^{-1}(V)$.*

Theorem 3.10. *If $f : X \to Y$ is gβ-continuous and aβ-continuous, then f is weakly contra-β-continuous.*

Proof. Assume $A \subseteq V \subseteq Y$, where A is closed in Y and V is open in Y. Since f is gβ-continuous, $f^{-1}(A)$ is gβ-closed. Thus, because $f^{-1}(A) \subseteq f^{-1}(V)$ and f is aβ-continuous, $\beta Cl(f^{-1}(A)) \subseteq f^{-1}(V)$, which proves that f is weakly contra-β-continuous.

Theorem 3.11. *If $f : X \to Y$ is weakly contra-β-continuous and images of gβ-closed sets are closed, then f is aβ-continuous.*

Proof. Let V be an open subset of Y, and let A be a gβ-closed subset of X such that $A \subseteq f^{-1}(V)$. Then $f(A)$ is closed, and $f(A) \subseteq V$. Since f is weakly contra-β-continuous, $\beta Cl(f^{-1}(f(A))) \subseteq f^{-1}(V)$. Then we have $\beta Cl(A) \subseteq \beta Cl(f^{-1}(f(A))) \subseteq f^{-1}(V)$, which proves that f is aβ-continuous.

4. STRONGLY Sβ-CLOSED SETS

Thompson [19] defines a spaces to be S-closed provided that every cover of the space by semi-open sets has a finite subfamily, the closures of whose members cover the space. From [9] a space is S-closed if and only if every cover of the space by regular closed sets has finite subcover. Dontchev [8] defines a space to be strongly S-closed provided that every cover of the space by closed sets has a finite subcover. Here we investigate the analogous definitions for β-closed sets.

Definition 13. *A space X is said to be nearly-compact* [18] *if every open cover of X has a finite subfamily such that the interiors of its closures cover X or equivalently if every cover of X by regular open sets has a finite subcover.*

The following characterization of near-compactness will be useful.

Lemma 4.1. *A set U is α-open if and only if $\beta Cl(U) = Int(Cl(Int(U)))$.*

Theorem 4.2. *A space X is nearly-compact if and only if every cover of X by α-open sets has a finite subfamily, the β-closures of whose members cover X.*

Proof. Assume X is nearly-compact. Let C be a cover of X by α-open sets. Then for every $U \in \mathcal{C}$, $U \subseteq Int(Cl(Int(U)))$. Therefore $\{Int(Cl(Int(U))) : U \in \mathcal{C}\}$ is a cover of X by regular open sets and hence has a finite subcover $\{Int(Cl(Int(U_i))) : i = 1, ..., n\}$. It then follows from Lemma 4.1 that $\{\beta Cl(U_i) : i = 1, ..., n\}$ is a cover of X.

Assume that every cover of X by α-open sets has a finite subfamily, the β-closures of whose members cover X. Let $\mathcal{C}$ be a cover of X by regular open sets. Then, since $\mathcal{C}$ is a cover of X by α-open sets, there exists a finite subfamily $\{U_i : i = 1, ..., n\}$ such that $\{\beta Cl(U_i) : i = 1, ..., n\}$ is a cover of X. However, since regular open sets are β-closed, $\beta Cl(U_i) = U_i$. Hence $\{U_i : i = 1, ..., n\}$ is a finite subcover of $\mathcal{C}$, which proves that X is nearly-compact.

Definition 14. *A space X is said to be strongly Sβ-closed if every cover of X by β-closed sets has a funite subcover.*

Theorem 4.3. *If X is strongly Sβ-closed, then X is nearly-compact.*

Proof. Let $\mathcal{C}$ be a cover of X by α-open sets. Then for every $U \in \mathcal{C}$, $U \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U)))$. Therefore $\{\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U))) : U \in \mathcal{C}\}$ is a cover of X. It then follows from Lemma 4.1 that $\{\beta\mathrm{Cl}(U) : U \in \mathcal{C}\}$ is a cover for X by β-closed sets. Since X is strongly Sβ-closed, there exists a finite subcover $\{\beta\mathrm{Cl}(U_i) : i = 1,..., n\}$ and hence by Theorem 4.2 X is nearly compact.

Remark 4.4. *Our original intention was to use the condition given in Theorem 4.2 as a definition of Sβ-closed. However, this condition turned out to be equivalent to the already defined notion of near-compactness.*

We have the following implications:

strongly Sβ-closed ⇒ strongly S-closed ⇒ S-closed

⇓

nearly-compact

The following examples show that none of the above implications are reversible.

Example 4.5. *Let X denote the real numbers with the topology* $\tau = \{U \subseteq X : 0 \notin U \text{ or } U = X\}$. *Since* $\{\{x, 0\} : x \neq 0\}$ *is a cover of X by regular closed sets with no finite subcover, X fails to be S-closed. However, since X is compact, X is also nearly-compact.*

Example 4.6. *Let X denote the real numbers with the indiscrete topology. Obviously X is strongly S-closed. However, since every subset of X is β-closed, X is not strongly Sβ-closed.*

Finally, Dontchev [8] established that strongly S-closed is strictly stronger than S-closed.

Next we investigate the relationship between strong Sβ-closure and compactness. Since Dontchev [8] showed that strong S-closure is independent of compactness, it follows that compactness does not imply strong Sβ-closure. The following example shows strong Sβ-closure is, indeed, independent of compactness.

Example 4.7. *Let X denote the real numbers with the topology* $\tau = \{U \subseteq X : 0 \in U \text{ or } U = \emptyset\}$. *Since the β-closed sets coincide with the closed sets, X is strongly Sβ-closed. However, X is not compact, since the open cover* $\{\{x, 0\} : x \neq 0\}$ *has no finite subcover.*

Theorem 4.8. *If X is strongly Sβ-closed, there exists a finite set S such that* $X = \beta\mathrm{Cl}(S)$.

Proof. The collection $\{\beta\mathrm{Cl}(\{x\}) : x \in X\}$ is a cover of X by β-closed sets. Since X is strongly Sβ-closed, there exists a finite subcover $\{\beta\mathrm{Cl}(\{x_i\}) : i = 1, ..., n\}$. Then $X = \cup_{i=1}^{n} \beta\mathrm{Cl}\{x_i\} \subseteq \beta\mathrm{Cl}(\{x_1, x_2,...,x_n\})$.

If we add the hypothesis that no singleton set of X be open to Theorem 4.8, then the singleton sets are β-closed and we obtain the following result.

Theorem 4.9. *A strongly Sβ-closed space in which the singleton sets are not open is finite.*

The proof of the following theorem is straightforward.

Theorem 4.10. *If $f : X \to Y$ is contra-β-continuous and X is strongly Sβ-closed, then $f(X)$ is compact.*

Definition 15. *A space X called a C-space provided that, for every open set U of X and every $x \in U$, there exists a closed set A such that $x \in A \subseteq U$.*

Theorem 4.11. *Let $f : X \to Y$ be weakly contra-β-continuous, and let Y be a C-space. If X is strongly Sβ-closed, then $f(X)$ is compact.*

Proof. Let $\mathcal{C}$ be an open cover of $f(X)$ by open subsets of X. Let $y \in f(X)$ and $V_y \in \mathcal{C}$ such that $y \in V_y$. Since Y is a C-space, there exists a closed set A_y such that $y \in A_y \subseteq V_y$. Because f is weakly contra-β-continuous, $\beta Cl(f^{-1}(A_y)) \subseteq f^{-1}(V_y)$. Then we see that $\{\beta Cl(f^{-1}(A_y)) : y \in f(X)\}$ is a cover of X by β-closed sets. Since X is strongly Sβ-closed, there exists a finite subcover $\left\{\beta Cl\left(f^{-1}\left(A_{y_i}\right)\right) : i = 1, 2, \ldots, n\right\}$. It follows that $\left\{V_{y_i} : i = 1, 2, \ldots, n\right\}$ is a finite subcover of $\mathcal{C}$ and hence that $f(X)$ is compact.

Definition 16. *A function $f : X \to Y$ is said to be contra-α-continuous if, for every open set V of Y, $f^{-1}(V)$ is α-closed.*

Theorem 4.12. *Let $f : X \to Y$ be weakly contra-β-continuous and contra-α-continuous and assume that Y is a C-space. If X is nearly-compact, then $f(X)$ is compact.*

Proof. Let $\mathcal{C}$ be an open cover of $f(X)$ by open subsets of X and let $y \in f(X)$. Then let $V_y \in \mathcal{C}$ such that $y \in V_y$. Since Y is a C-space, there exists a closed set A_y such that $y \in A_y \subseteq V_y$. Since f is contra-α-continuous, $f^{-1}(A_y)$ is α-open. It follows that the family $\{f^{-1}(A_y) : y \in f(X)\}$ is a cover of X by α-open sets. Since Y is nearly-compact, by Theorem 4.2 there exists a finite subfamily $\left\{f^{-1}\left(A_{y_i}\right) : i = 1, \ldots, n\right\}$ such that $X = \cup_{i=1}^{n} \beta Cl\left(f^{-1}\left(A_{y_i}\right)\right)$.

Because f is weakly contra-β-continuous, $\beta Cl\left(f^{-1}\left(A_{y_i}\right)\right) \subseteq f^{-1}\left(V_{y_i}\right)$ for every i. Thus $\left\{V_{y_i} : i = 1, \ldots, n\right\}$ is a finite subcover of $\mathcal{C}$ and we see that $f(X)$ is compact.

Remark 4.13. *The combination of weak contra-β-continuity and a C-space for a codomain does not imply contra-β-continuity. Consider the identity mapping on the real numbers with the usual topology.*

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