Jour. Pure Math., Vol. 24, 2007, pp. 39-47

CROSSING AND THICKNESS OF SPECIAL TYPE OF NON-PLANAR GRAPH

ANUPAM DUTTA, BICHITRA KALITA, HEMANTA K. BARUAH

ABSTRACT : In this paper, we have constructed some non-planar graphs from the TEEP graph and studied various properties of them, relating to the thickness and crossing. In addition to this, the edge-disjoint Hamiltonian circuit of the non-planar graphs has been discussed.

AMS Subject-Classification: 05C30, 05C45, 05c62, 05c83 (MSC 2000)

1. INTRODUCTION

The thickness and crossings of non-planar graph play very important role in case of printed circuit board. It is important to minimize the wire crossings in case of VLSI design technology, particularly in circuit layout process. There are graphs, namely Hyper graph, clique graph, interval graph, circuit graph etc. which are directly or indirectly related with VLSI design in floor plan. [12]. Many problems in this area are still remaining unsolved. It is known that the thickness and crossing of a planar graph is one and zero respectively. The thickness and crossings of a few non-planar graphs have been found [1-7]. Recently Kalita B, [8] found a set of non-planar graph, and studied the maximum and minimum number of crossings. He further cited some examples with corssings of non-planar graph. The upper bound of crossing of complete graph Km and bipartite graph Km, n is found by the theorem (11.23) [13]. The problem of finding the crossings and thickness of an arbitrary non-planar graph is not found till today. There are some particular types of graphs in which crossings and thickness can be found.

In this paper, we have developed some non-planar graphs and studied the crossings and thickness of them. The non-planar graphs have been constructed from TEEP graph [8,14]. The paper is organized as follows. In section 1, we have explained some works of crossing and thickness of graph. In section 2, the notation and terminology are considered. The definition and construction of non-planar graphs are included in section 3. Section 4 contains the theoretical explanation and properties. The conclusion is included in section 5.

2. NOTATIONS AND TERMINOLOGY

The notation and terminology are considered from the standard reference [1-14]. The thickness and crossings of non-planar graph G are denoted by θ (G) and Φ (G). The maximum and minimum degrees are denoted by Δ and ∂ respectively.

3. DEFINITION AND CONSTRUCTION OF NON-PLANAR GRAPH

Before going to construct the non-planer graph from TEEP graphs, we remind the definition of crossings and thickness of non-planar graph.

3.1. Crossing. The minimum number of crossing in a drawing of a non-planar graph G in a plane is called the crossing number. It is denoted by Φ (G).

3.2. Thickness. The thickness of a non-planar graph G is the minimum number of planar graphs in a decomposition of G into planar sub-graphs graphs. It is denoted by θ (G).

3.3. Construction of non-planar graph from TEEP graph. Let G (2m + 2,6m) and H (2m+3, 6m+3) for m ≥ 2 are two TEEP graphs [14]. These two graphs have minimum degree δ = 3 and maximum degree $\Delta = 2m + 1$ and $\Delta = 2m + 2$ for $m \geq 2$ respectively. There are two vertices of degree four each, two vertices of degree three and the remaining vertices are of degree five. The following two figures [fig 1 & fig 2] give the TEEP graph G & H for m = 2.

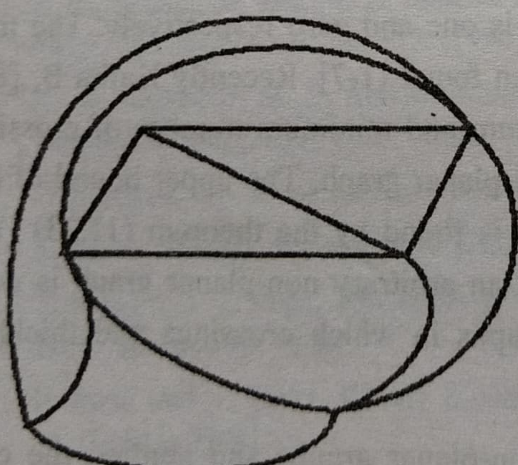

fig-1

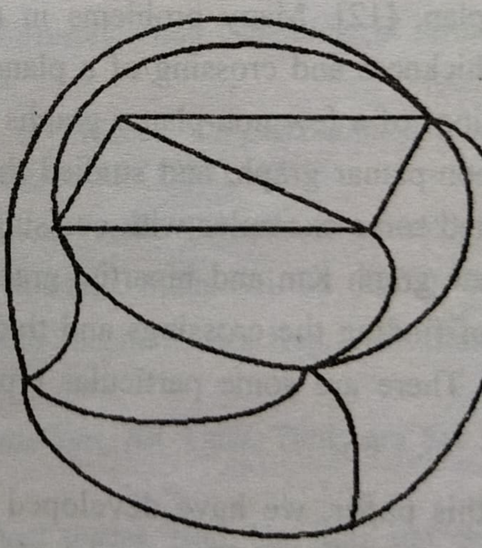

fig-2

Now joining the two vertices of minimum degree by one edge of the graph G (2m + 2, 6m) for m ≥ 2 [similarly for the graph H (2m + 3,6m + 3) for m ≥ 2], and continuing the process of joining the other vertices of minimum degree of the graph G (2m + 2,6m)/ H (2m + 3, 6m + 3) for m ≥ 2 by one edge till the graph thus obtained forms a non-planar graph which is the preceding to the complete graph K_{2m-2}/K_{2m+3} for m ≥ 2. Thus, we have two non-planar graph G′(2m + 2, 6m + 2) and H′ (2m + 3, 6m + 8) for m ≥ 2. The figure (3) and (4) shows the non-planar graphs G′ (6.14) and H′ (7, 20) for m = 2 which are the preceding one of the complete graph K_{2m+2}/K_{2m+3}.

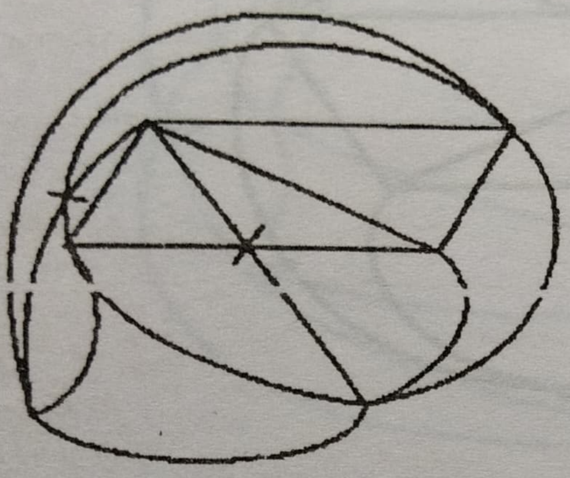

fig-3

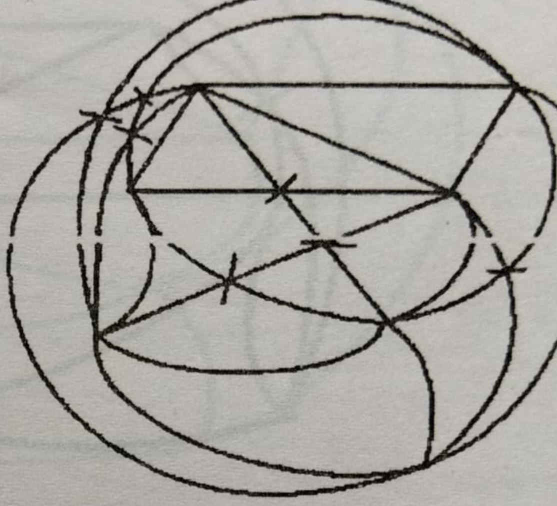

fig-4

These two graphs do not satisfy the conditions of planarity that is the number of edges is not les than or equal to 3n – 6, where n is the number of vertices of the graph. Hence we have the following two inequalities from the construction process of the non-planar graphs drawn earlier as

$$6m + 2 \not\leq 3(2m + 2) - 6 \text{ and}$$

$$6m + 8 \not\leq 3(2m + 3) - 6 \text{ for } m \geq 2.$$

Remarks: The non-planar graphs constructed in this paper as shown earlier in figure –3, figure –4, which is completely different types of non-planar graph drawn by Kalita [8].

4. THEORETICAL EXPLANATION AND PROPERTIES

4.1. Theorem. For the non-planar graph G′(2m + 2, 6m + 1) for m = 2, 3, 4... the number of crossings is m – 1 that is Φ(G) = m – 1.

Proof. We know that the TEEP graph (2m + 2, 6m) for m ≥ 2 [14] is a planar graph. WE have constructed the graph G′(2m + 2, 6m + 1) for m ≥ 2 by joining the vertices of minimum degree δ = 3 of the graph G(2m + 2, 6m) by one edge. Definitely the G′(2m + 2, 6m + 1) for m ≥ 2 is a non-planar as we have

$$6m + 1 \not\leq 3(2m + 2) - 6$$

$$\Rightarrow 6m + 1 \not\leq 6m \text{ for } m \geq 2.$$

Now we have to prove that the number of crossings of the graph $G'(2m + 2, 6m + 1)$ is $(m - 1)$, $\Phi(G') = m - 1$.

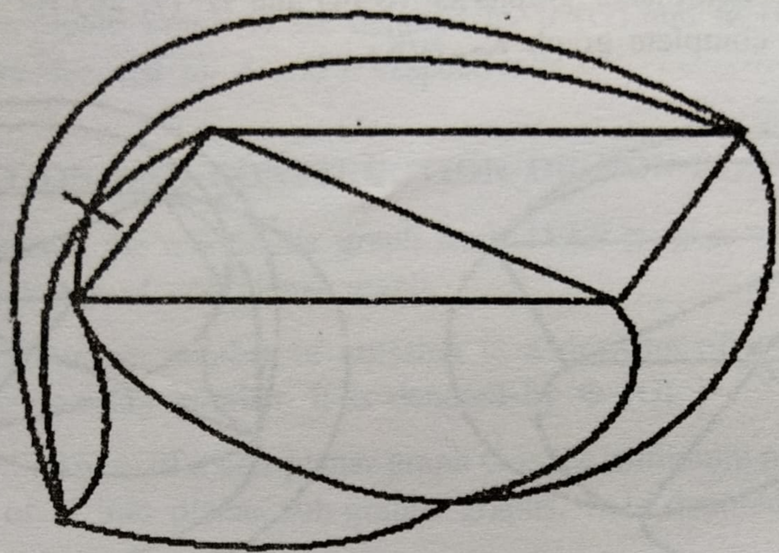

fig-5

Let us prove it by the method of induction. It is observed that for $m = 2$ the crossings number is one. [From fig-5] $\Phi(G') = 1$. Which is true for $m = 2$.

Now for $m = 3$ it can be shown that the number of crossings is two.

Let us consider that the above theorem is true for $m = K$, where the number of crossings is $K - 1$. I.e. the crossings of $G'(2K + 2, 6K + 1)$ is $(K - 1)$. For the next stage we proceed by adding two vertices and six edges we have the structure $G'(2K + 2 + 2, 6K + 1 + 6)$ and the number of crossings are $(K - 1 + 1)$ i.e. $G'[2(K + 1) + 2, 6(k + 1) + 1]$ the crossing number is K. Hence the theorem is true for $m = K + 1$. i.e. the number of crossings for $m = K + 1$ is K. Which is true for $m = K + 1$. Hence the theorem is true for all $m \geq 2$. Now we can conclude that the number of crossings of non-planar graph $G'(2m + 2, 6m + 1)$ for $m \geq 2$ is $(m - 1)$.

I.e. Φ $(G) = m - 1$.

4.2. Theorem. The crossings of non-planar graph $H'(2m + 3, 6m + 4)$ for $m \geq 2$ is m i.e. $\Phi(H') = m$.

Proof. We have for the TEEP graph $H(2m + 3, 6m + 3)$ for $m \geq 2$, the minimum degree $\delta = 3$ and maximum degree $\Delta = 2m + 2$. There are two vertices of minimum degree $\partial = 3$ and two vertices of degree four and the remaining vertices are of degree five. Now we have to prove that the number of crossings of the graph $H'(2m + 3, 6m + 4)$ for $m \geq 2$ is m.

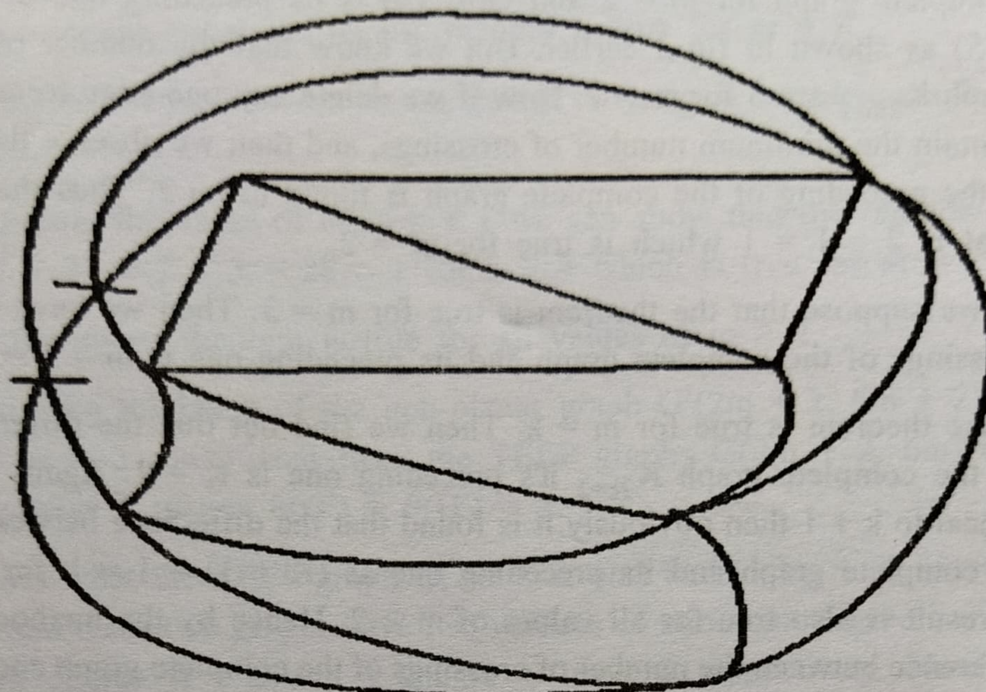

fig. 6

Let us prove it by the method of induction. It is observed that for m = 2 the graph H′(7, 16) has two crossings [fig- 6]. Again it can be shown that for m = 3 the graph H′(9, 22) has three crossing. Thus we found that the result is true for m = 2 and m = 3.

Let us consider that the above theorem is true for m = k. Then the structure of the non-planer graph is H′(2k + 3, 6k + 4) and then the number of crossing is K. Now when we put m = k + 1 then the non-planar graph H′(2K + 3, 6K + 4) takes the from as H′(2K + 5, 6K + 4) for K ≥ 3, Which automatically shows that the crossings of the graph is k + 1. Hence the above theorem is true for all m ≥ 2 i.e. the of crossings of the non-planar graph H′(2m + 3, 6m + 4) for m ≥ 2 are m i.e. $\Phi(H') = m$.

4.3. Theorem. The difference between the number of crossings of the complete graph K_{2m+2} for m ≥ 2 and the number of crossings of its preceding one (not complete) constructed from the TEEP graph G(2m + 2, 6m) for m ≥ 2 is ≥ (m – 1).

Proof. We have the TEEP graph G(2m + 2, 6m) for m ≥ 2 is a planar graph where the two vertices has minimum degree $\delta = 3$ other two vertices has degree four, one vertex has maximum degree $\Delta = 2m + 1$ but remainging vertices are of degree five as shown earlier in fig. (1) & (2). Now we have to prove the difference between the crossings of complete graph K_{2m+2} and its preceding one is always ≥ (m – 1).

Let us prove it by the method of induction. Now for m = 2, the structure of G(2m + 2, 6m) is of the form G(6, 12) which is clearly a planar graph. But we have

G(6, 15) is a complete graph for m = 2 and G(6, 14) is its preceding one of the complete graph of G(6, 15) as shown in fig 3 earlier. But we know that the number of crossings of the complete graph K_{2m+2} are 3 for m = 2. Now if we delete any one edge from the complete graph, which contain the minimum number of crossings, and then we observe that the number of crossings of the preceding of the complete graph is found to be 2. Thus the difference is (3 – 2) = 1, that is 2 – 1 = 1 which is true for m = 2.

Similarly we suppose that the theorem is true for m = 3. Then we have the difference between the crossings of the complete graph and its preceding one is m – 1 = 2 for m = 3.

Suppose the theorem is true for m = k. Then we find out that the difference between the crossing of the complete graph K_{2k+2} it's preceding one is K – 1. Again if we put the value of m is equal to k + 1 then obviously it is found that the difference between the number of crossings of complete graph and its preceding one as (K + 1) – 1 = k for k ≥ 3 which shows that the result is also true for all values of m ≥ 2. Hence by the method of induction we have the difference between the number of crossings of the complete graph and its preceding one is always ≥ (m – 1).

4.4. Theorem. The difference between the number of crossings of complete graph K_{2m+3} and its preceding one (constructed from the TEEP graph H(2m + 3, 6m + 3) of the complete graphs for m ≥ 2 is at least (2m – 3).

Proof. We know that the TEEP graph H(2m + 3, 6m + 3) for m ≥ 2 is a planar graph. We have construected earlier a non-planar graph as shown in the [fig-4], which is the preceding one of the complete graph K_{2m+3}.

Now we are to prove that the difference between the number of crossings of the complete graph K_{2m+3} and its preceding one is (2m – 3).

We will prove it by the method of induction.

Now for m = 2 the structure H(7, 15), which is nothing but a TEEP graph, and it is a planar graph and H(7, 21) is the complete graph of the above planar graph. Now joining the vertices by one edge in between the two minimum degree vertices continuously till the preceding one of the complete graph is obtained, then we observed that the number of crossings of the preceding one is 8. But, the crossings of the complete graph K_{2m+3} are 9.

Thus we observed that thedifference between the number of crossings of complete graph and its preceding one is (9 – 8) = 1 i.e. (2.2 – 3) = 1 which shows that the theorem is true for m = 2.

Again if we put the value of m equal to 3 then the number of crossings of the complete graph and the number of crossing of preceding complete graph is (2.3 – 3) = 3.

Now we consider that the above theorem is true for m = K.

Hence the difference between the number of crossing of complete graph and its preceding one is 2K – 3.

Now, putting the value of m as k + 1 we can show that the relation 2m – 3 gives as 2(k + 1) – 3 = 2k + 2 – 3 = 2k – 1 for k ≥ 3 which is true for m = k + 1.

Hence the above theorem is true for all values of m ≥ 2.

4.5. Theorem. The thickness of the non-planar graph G′(2m + 2, 6m + 2) and H′(2m + 3, 6m + 8) for m ≥ 2 constructed from the TEEP graphs G(2m + 2, 6m) and H (2m + 3, 6m + 3) is always 2 that is θ(G′) = θ(H′) = 2.

Proof. We know that the TEEP graph G(2m + 2, 6m) and H(2m + 3, 6m + 3) for m≥2 are planar graph and these two graphs have two vertices of minimum degree δ = 3 and one vertex has maximum degree A = 2m + 1 and A = 2m + 2 respectively. Now from the graph G(2m + 2, 6m) and H (2m + 3, 6m + 3) for m ≥ 2 by joining vertices of minimum degrees by one edge only and continuiting the process till the preceding of complete graphs G′(2m + 2, 6m + 2) and H′(2m + 3, 6m + 8) are obtained and they are non-planar. But we require finding out the decomposition of these two non-planar graphs as planar sub-graphs and it can be shown that there will be only two such decompositions for theses two non-planar graphs. That is the thickness of the non-planar graphs G′(2m + 2, 6m + 2) and H′(2m + 3, 6m + 8) is always two for m = 2.

Now for m = 3 we observe that the thickness of the graphs G′(2m + 2, 6m + 2) and H′(2m + 2, 6m + 5) is also 2.

Thus we can conclude that the above theorem is true for m = 2 and m = 3.

We know that thickness of the complete graphs K_{2m+2} and K_{2m+3} for m ≥ 2 can be found from the theorems [11.23 and 11.24] [13] and we have that the non-planar graphs G′(2m + 2, 6m + 2) and H′(2m + 3, 6m + 8) for m ≥ 2 are constructed from the TEEP graphs G(2m + 2, 6m) and H(2m + 3, 6m + 3) for m ≥ 2 by joining some vertices by some edges as discussed in section 3.3. It can be shown easily that there requires only two plans to minimize the number of crosing of these two graphs for m ≥ 2. Hence we must comment that the non-planar graph G′(2m + 2, 6m + 2) and H′(2m + 3, 6m + 8) have require only two planes. That is the thickness is two.

4.6. Thorem. The number of edge disjoint Hamiltonian circuits of the non-planar graph G′(2m + 2, 6m + 2) made from the planar graph G(2m + 2, 6m) for m ≥ 2 up to the preceding of K_{2m+2} is always 2.

Proof. We know that from the TEEP graph G(2m + 2, 6m) for m > 2 which has only one edge joint Hamiltonian Circuit. Now if we joined by one edge in between the minimum degree vertices by one edge and continue the process up to the preceding of complete graph of the mentioned graph then also the edge disjoint Hamiltonian Circuit is also two [from fig (3)]. Because for m = 2 the degree of the vertices up to the preceding one of the complete graph is observed that which has four vertices of maximum degree $\Delta = 5$ and two vertices of minimum degree $\delta = 4$. So it has only two edges disjoint Hamiltonian Circuits.

Which is true for m = 2

Thus trivially we can proved that for all values of $m \geq 2$ the non-planar graph G′(2m + 2, 6m + 1) for $m \geq 2$ which has two edge disjoint Hamiltonian circuits. And continuing the process of joining the vertices by edges up to proceding of the complete graph K_{2m+2} for $m \geq 2$, which has also two edges, disjoint Hamiltonian Circuits.

4.7. Theorem. The number of edge disjoint Hamiltonian Circuits of the non-planar graph made from the planar graph H (2m + 3, 6m + 3) for $m \geq 2$ up to the preceding of K_{2m+3} is always 2.

Proof. The proof of this theorem as [4.6].

5. CONCLUSION

This paper gives a special type of non-planar graphs, which are constructed from the TEEP graphs and the thickness and crossing of these non-planar graphs are found from various theorems. Hence one can study the thickness and crossing of these non-planar graphs for various design purposes, which is lies in the VLSI design technology.

REFERENCES

1. Alekseev, V. B. and V. S. Goncakov, "The thickness of an arbitrary complete Graph" *Mat. Sb (N.S.) 101 (143)*, (P.P. 221-230), 1976.

2. Beineke, L, W., F. Harary, "On the thickness of complete graph" *Canad. J. Math 17*, (PP 850-859), 1965.

3. Erdos. P and R. K. Guy, "Crossing number problems" *Amer. Math. Monthly 80*, (PP 52-58), 1973.

4. Guy. R. K., "Crossing numbers of graphs" *In graph theorey & Appl. Kalamazoo, (ed. Y. Alavieletal), Lect. Notes, Maths, 303 Springer*, (PP 111-124), 1972.

5. Kleitman. D. J., "The Crossing number of $K_{5.n}$." *J. Comb. Jh. 9* (PP 315-323), 1970.

6. Pach. J., and G. Toth, "graphs drawn with few crossings per edges" *Combinatorica 17,* (PP 427-439), 1997.

7. Szekely L. A., "Crossing numbers and hard Erdos problem in discrete geometry" *Combin. Prabab comput 6* (PP 353-358), 1997.

8. Kalita, B. "A new set of non planar graphs" *Bull of Pure & Appl Sc., Vol 24E (No. 1)* (PP 29-38), 2005.

9. Barnes E. R., "An algorithm for portioning the nodes of a graph" *Technical Report IBM, T.J. waston research center Dept. Computer Science,* 1981.

10. Bhaskar T., S. Sahni; "A linear algorithm to find a rectangular dual of a planar Triangulated graph" *Algorithanica 3 (2)* (PP 274-278), 1988.

11. Bondy J. A., and U.S.R. Murty, "Graph theory with applications", *American Elsevier, New York,* 1976.

12. Kozminski. K., and Kinnen, "Rectangular dual of planar graphs" *Networks 15,* (PP 116-157) 1985.

13. F. Harary, "Graph theory" *Narosa publishing House* (PP 120-123) 1995.

14. Kalita B, "Some investigations on graph theory" *Ph.D. thesis, Finance India Vol. XIX, No-4 Dec. (P.P. 1430-1438), 2005.*

15. Kalita. B. "A few unsolved problems of graph theory", *Proc. international conference of Information technology held at Haldia, India (p.p. 564-567),* 2007.

Anupam Dutta
Patidarrang College, Muktapur.
P.O. Loch, Kamrup, Assam, India.
Email-anupa.dutta@gmail.com

Bichitra Kalita
Department of Computer Science
Assam Engineering College
Guwahati-13, Assam, India.
Email-bichitral_kalita@rediff.com

Hemanta. K. Baruah.
Department of Statistics.
Gauhati University
Guwahati-14, Assam, India.
Email-hemanta_bh@yahoo.com