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TRANSFORMATIONS OF SOME MULTIPLE HYPER-GEOMETRIC FUNCTIONS OF SEVERAL VARIABLES*

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ABSTRACT : In the present paper, the author applies the difference operators, to establish many transformations of multiple hypergeometric functions of several variables introduced by Chandel et al. ([4] and [5]). He also discusses their special cases.

Key words and pharases : Multiple Hypergeometric Functions of Several Variables, Difference Operators, Transformations.

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1. INTRODUCTION

Making an appeal to the difference operators Δ and E, Agrawal [1, 2] and Gupta and Agrawal [7, 8, 9] obtaned various transformations of hypergeometric series. Further, Gupta and Agarwal [10] obtained a transformation of Lauricella's multiple hypergeometric function $F_A^{(n)}$ ([11]; see also [13]).

In the present paper, making an appeal to these difference operators, we obtain various transformations of some multiple hypergeometirc functions of several variables introduced and studied by Chandel et al. ([3] to [5]). We also discuss their special cases, which may be useful in related topics of mathematical analysis.

2. TRANSFORMATION FORMULAS

Consider

$$A = \frac{\Gamma(c_1+\ldots+c_k)\Gamma(c_{k+1})\ldots\Gamma(c_n)}{\Gamma(b_1)\ldots\Gamma(b_n)}[1 - x_1E_1 - \ldots - x_nE_n]^{-a}$$

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$$\left\{\frac{\Gamma(b_1)\ldots\Gamma(b_n)}{\Gamma(c_1+\ldots+c_k)\Gamma(c_{k+1})\ldots\Gamma(c_n)}\right\}$$

$$= \frac{\Gamma(c_1+\ldots+c_k)\Gamma(c_{k+1})\ldots\Gamma(c_n)}{\cdot\Gamma(b_1)\ldots\Gamma(b_n)} \sum_{m_1,\ldots,m_n=0}^{\infty} (a)_{m_1+\ldots+m_n} \frac{x_1^{m_1}}{m_1\,!}\cdots\frac{x_n^{m_n}}{m_n\,!}$$

$$\left\{\frac{\Gamma(b_1+m_1)\ldots\Gamma(b_n+m_n)}{\Gamma(c_1+\ldots+c_k+m_1+\ldots+m_k)\Gamma(c_{k+1}+m_{k+1})\ldots\Gamma(c_n+m_n)}\right\}$$

$$= {}_{(k)}F_{AD}^{(n)}(a, b_1, \ldots, b_n, b_n, c_1+\ldots+c_k, c_{k+1}, \ldots, c_n, x_1, \ldots, x_n)$$

Also we can write

$$A = \frac{\Gamma(c_1+\ldots+c_k)\Gamma(c_{k+1})\ldots\Gamma(c_n)}{\Gamma(b_1)\ldots\Gamma(b_n)}\left[1-x_{k+1}-\ldots-x_n\right]^{-a}$$

$$\left[1-\frac{x_1E_1+\ldots+x_kE_k+x_{k+1}\Delta_{k+1}+\ldots+x_n\Delta_n}{1-x_{K+1}-\ldots-x_n}\right]^{-a}\left\{\frac{\Gamma(b_1)\ldots\Gamma(b_n)}{\Gamma(c_1+\ldots+c_k)\Gamma(c_{k+1})\ldots\Gamma(c_n)}\right\}$$

$$= (1-x_{k+1}-\ldots-x_n)^{-a}$$

$${}_{(k)}F_{AD}^{(n)}\Bigg(a, b_1, \ldots, b_k, c_{k+1}-b_{k+1}, \ldots, c_n-b_n; c_1+\ldots+c_k, c_{k+1}, \ldots c_n$$

$$\frac{x_1}{1-x_{k+1}-\ldots-x_n}, \ldots, \frac{x_k}{1-x_{k+1}-\ldots-x_n}, \frac{-x_{k+1}}{1-x_{k+1}-\ldots-x_n}, \ldots, \frac{-x_n}{1-x_{k+1}-\ldots-x_n}\Bigg).$$

Equating both values of A, we derive

$$(2.1)\quad {}^{(k)}F_{AD}^{(n)}\left(a,b_1,\ldots,b_n;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right)$$

$$=\left(1-x_{k+1}-\ldots-x_n\right)^{-a}$$

$${}^{(k)}F_{AD}^{(n)}\left(a,b_1,\ldots,b_k,c_{k+1}-b_{k+1},\ldots,c_n-b_n;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right)$$

$$\frac{x_1}{1-x_{k+1}-\ldots-x_n},\ldots,\frac{x_k}{1-x_{k+1}-\ldots-x_n},\frac{-x_{k+1}}{1-x_{k+1}-\ldots-x_n},\ldots,\frac{-x_n}{1-x_{k+1}-\ldots-x_n}\Bigg)$$

where ${}^{(k)}F_{AD}^{(n)}$ is multiple hypergeometric function related to Lauricella's functions introduced by Chandel and Gupta [4].

Similarly, by considering

$$\frac{\Gamma(c_1)\ldots\Gamma(c_n)}{\Gamma(b_1+\ldots+b_k)\Gamma(b_{k+1})\ldots\Gamma(b_n)}\left[1-x_1E_1-\ldots-x_nE_n\right]^{-a}$$

$$\left\{\frac{\Gamma(b_1+\ldots+b_k)\Gamma(b_{k+1})\ldots\Gamma(b_n)}{\Gamma(c_1)\ldots\Gamma(c_n)}\right\},$$

we derive

$$(2.2)\quad {}^{(k)}F_{AC}^{(n)}\left(a,b,b_{k+1},\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n\right)$$

$$=\left(1-x_{k+1}-\ldots-x_n\right)^{-a}\,{}^{(k)}F_{AC}^{(n)}\Bigg(a,b,c_{k+1}-b_{k+1},\ldots,c_n-b_n;c_1,\ldots,c_n;$$

$$\frac{x_1}{1-x_{k+1}-\ldots-x_n},\ldots,\frac{x_k}{1-x_{k+1}-\ldots-x_n},\frac{-x_{k+1}}{1-x_{k+1}-\ldots-x_n},\ldots,\frac{-x_n}{1-x_{k+1}-\ldots-x_n}\Bigg).$$

For $k=0$, (2.1) and (2.2) both give following transformation formula for Lauricella's multiple hypergeometric function $F_A^{(n)}$ [11].

(2.3) $F_A^{(n)}\left(a,b_1,\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n\right)$

$$= \left(1-x_1-\ldots-x_n\right)^{-a} F_A^{(n)}\left(a,c_1-b_1,\ldots,c_n-b_n;c_1,\ldots,c_n;\frac{-x_1}{1-x_1-\ldots-x_n},\ldots,\frac{-x_n}{1-x_1-\ldots-x_n}\right),$$

which can be verified by choosing $k = 2$ separately in both the results due to Agrawal [1, p. 116] and Gupta and Agrawal [10].

Further, for $k = 1$, both give

(2.4) $F_A^{(n)}\left(a,b_1,\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n\right)$

$$= \left(1-x_2-\ldots-x_n\right)^{-a} F_A^{(n)}\left(a,c_1-b_1,\ldots,c_n-b_n;c_1,\ldots,c_n;\frac{x_1}{1-x_2-\ldots-x_n},\frac{-x_2}{1-x_2-\ldots-x_n},\ldots,\frac{-x_n}{1-x_2-\ldots-x_n}\right),$$

which can be further extended in the from of n unified results

(2.5) $F_A^{(n)}\left(a,b_1,\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n\right) = \left(1-x_1-\ldots-x_{i-1}-x_{i+1}-\ldots-x_n\right)^{-a}$

$$F_A^{(n)}\left(a,c_1-b_1,\ldots,c_{i-1}-b_{i-1},b_i,c_{i+1}-b_{i+1},\ldots,c_n-b_n;c_1,\ldots,c_n;\right.$$

$$\frac{-x_1}{1-x_1-\ldots-x_{i-1}-x_{i+1}-\ldots-x_n},\ldots,\frac{-x_{i-1}}{1-x_1-\ldots-x_{i-1}-x_{i+1}-\ldots-x_n},$$

$$\frac{x_i}{1-x_i-\ldots-x_{i-1}-x_{i+1}-\ldots-x_n},\frac{-x_{i+1}}{1-x_1-\ldots-x_{i-1}-x_{i+1}-\ldots-x_n}$$

$$\left.\ldots,\frac{-x_n}{1-x_1-\ldots-x_{i-1}-x_{i+1}-\ldots-x_n}\right),\ i = 1,\ 2,\ \ldots,\ n.$$

Further considering

$$\frac{\Gamma(c_1)\dots\Gamma(c_n)}{\Gamma(b_{k+1})\dots\Gamma(b_n)}\left[1-x_1E_1-\dots-x_nE_n\right]^{-a}\left\{\frac{\Gamma(b_{k+1})\dots\Gamma(b_n)}{\Gamma(c_i)\dots\Gamma(c_n)}\right\}$$

we derive

$$(2.6)\quad {}^{(k)}_{(2)}\phi^{(n)}_{AC}\left(a,b_{k+1},\dots,b_n;c_1,\dots,c_n;x_1,\dots,x_n\right)$$

$$=\left(1-x_{k+1}-\dots-x_n\right)^{-a}\,{}^{(k)}_{(2)}\phi^{(n)}_{AC}\left(a,c_{k+1}-b_{k+1},\dots,c_n-b_n;c_1,\dots,c_n;\right.$$

$$\left.\frac{x_1}{1-x_{k+1}-\dots-x_n},\dots,\frac{x_k}{1-x_{k+1}-\dots-x_n},\frac{-x_{k+1}}{1-x_{k+1}-\dots-x_n},\dots,\frac{-x_n}{1-x_{k+1}-\dots-x_n}\right),$$

where ${}^{(k)}_{(2)}\phi^{(n)}_{AC}$ is the confluent form of Karlsson's multiple hypergeometric function introduced by Chandel and Vishwakarma [5].

For $k = 0$. (2.6) further gives (2.3).

3. GENERALIZATION OF (2.1)

Equation (2.1) can also be written as follows:

$$A=\frac{\Gamma(c_1+\dots+c_k)\Gamma(c_{k+1})\dots\Gamma(c_n)}{\Gamma(b_1)\dots\Gamma(b_n)}\left[1-x_{k+1}-\dots-x_n\right]^{-a}$$

$$\left[1-\frac{x_1E_1+\dots+x_mE_m+x_{m+1}\Delta_{m+1}+\dots+x_n\Delta_n}{1-x_{m+1}-\dots-x_n}\right]^{-a}\left\{\frac{\Gamma(b_1)\dots\Gamma(b_n)}{\Gamma(c_1+\dots+c_k)\Gamma(c_{k+1})\dots\Gamma(c_n)}\right\}$$

$$=\left(1-x_{m+1}-\dots-x_n\right)^{-a}$$

$${}^{(k)}F^{(n)}_{AD}\left(a,b_1,\dots,b_m,c_{m+1}-b_{m+1},\dots,c_n-b_n;c_1+\dots+c_k,c_{k+1},\dots,c_n;\right.$$

$$\left.\frac{x_1}{1-x_{m+1}-\dots-x_n},\dots,\frac{x_m}{1-x_{m+1}-\dots-x_n},\frac{-x_{m+1}}{1-x_{m+1}-\dots-x_n},\dots,\frac{-x_n}{1-x_{m+1}-\dots-x_n}\right),$$

which gives

$$(3.1)\quad {}^{(k)}F_{AD}^{(n)}\left(a,b_1,\ldots,b_n;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right)$$

$$=\left(1-x_{m+1}-\ldots-x_n\right)^{-a}\,{}^{(k)}F_{AD}^{(n)}\Bigg(a,b_1,\ldots,b_m,c_{m+1}-b_{m+1},\ldots,c_n-b_n;c,c_{k+1},\ldots,c_n;$$

$$\frac{x_1}{1-x_{m+1}-\ldots-x_n},\ldots,\frac{x_m}{1-x_{m+1}-\ldots-x_n},\frac{-x_{m+1}}{1-x_{m+1}-\ldots-x_n},\ldots,\frac{-x_n}{1-x_{m+1}-\ldots-x_n}\Bigg)$$

where m and k are non-negative integers $\leq n$.

For $m = k$, (3.1) reduces to (2.1).

Replacing k by 0 and m by k, (3.1) reduces to the following result :

$$(3.2)\quad F_A^{(n)}\left(a,b_1,\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n\right)$$

$$=\left(1-x_{k+1}-\ldots-x_n\right)^{-a}F_A^{(n)}\Bigg(a,b_1,\ldots,b_k,c_{k+1}-b_{k+1},\ldots,c_n-b_n;c_1,\ldots c_n;$$

$$\frac{x_1}{1-x_{k+1}-\ldots-x_n},\ldots,\frac{x_k}{1-x_{k+1}-\ldots-x_n},\frac{-x_{k+1}}{1-x_{k+1}-\ldots-x_n},\ldots,\frac{-x_n}{1-x_{k+1}-\ldots-x_n}\Bigg),$$

which also suggets the following results due to Gupta and Agrawal [10] and Agrawal [2]:

$$(3.3)\quad F_A^{(n)}\left(a,b_1,\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n\right)$$

$$=\left(1-x_1-\ldots-x_n\right)^{-a}F_A^{(n)}\Bigg(a,c_1-b_1,\ldots,c_k-b_k,b_{k+1},\ldots,b_n;c_1,\ldots c_n;$$

$$\frac{-x_1}{1-x_1-\ldots-x_k},\ldots,\frac{-x_k}{1-x_1-\ldots-x_k},\frac{x_{k+1}}{1-x_1-\ldots-x_k},\ldots,\frac{x_n}{1-x_1-\ldots-x_k}\Bigg).$$

As for (2.2), again considering

$$
{}^{(k)}F_{AC}^{(n)}\left(a, b_1 + \ldots + b_k, b_{k+1}, \ldots, b_n; c_1, \ldots, c_n; x_1, \ldots, x_n\right)
$$

$$
= \frac{\Gamma(c_1)\ldots\Gamma(c_n)}{\Gamma(b_1 + \ldots + b_k)\Gamma(b_{k+1})\ldots\Gamma(b_n)}\left[1 - x_1 E_1 - \ldots - x_n E_n\right]^{-a}
$$

$$
\left\{\frac{\Gamma(b_1 + \ldots + b_k)\Gamma(b_{k+1})\ldots\Gamma(b_n)}{\Gamma(c_1)\ldots\Gamma(c_n)}\right\}
$$

$$
= \frac{\Gamma(c_1)\ldots\Gamma(c_n)}{\Gamma(b_1 + \ldots + b_k)\Gamma(b_{k+1})\ldots\Gamma(b_n)}\left[1 - x_{M+1} - \ldots - x_n\right]^{-a}
$$

$$
\left[1 - \frac{x_1 E_1 + \ldots + x_M E_M + x_{M+1}\Delta_{M+1} + \ldots + x_n \Delta_n}{1 - x_{M+1} - \ldots - x_n}\right]^{-a}
$$

$$
\left\{\frac{\Gamma(b_1 + \ldots + b_k)\Gamma(b_{k+1})\ldots\Gamma(b_n)}{\Gamma(c_1)\ldots\Gamma(c_n)}\right\}
$$

we establish

(3.4) $$
{}^{(k)}F_{AC}^{(n)}\left(a, b, b_{k+1}, \ldots, b_n; c_1, \ldots, c_n; x_1, \ldots, x_n\right)
$$

$$
= \left(1 - x_{M+1} - \ldots - x_n\right)^{-a}
$$

$$
{}^{(k)}F_{AC}^{(n)}\Bigg(a, b, b_{k+1}, \ldots, b_M; c_{M+1} - b_{M+1}, \ldots, c_n - b_n; c_1, \ldots, c_n;
$$

$$
\frac{x_1}{1 - x_{M+1} - \ldots - x_n}, \ldots, \frac{x_M}{1 - x_{M+1} - \ldots - x_n}, \frac{-x_{M+1}}{1 - x_{M+1} - \ldots - x_n}, \ldots, \frac{-x_n}{1 - x_{M+1} - \ldots - x_n}\Bigg),
$$

$(0 \le k \le M \le n$ and $k,\ M\ n \in I)$.

which, for $M = k$, reduces to (2.2); while for $M = k = 0$, (3.4) reduces to (2.3).

Further, for $k = 0$, (3.4) reduces to

(3.5) $$F_A^{(n)}\left(a,b_1,\ldots,b_n;\ c_1,\ldots,c_n;\ x_1,\ldots,x_n\right)$$

$$= \left(1-x_{M+1}-\ldots-x_n\right)^{-a} F_A^{(n)}\left(a,b_1,\ldots,b_M;\ c_{M+1}-b_{M+1},\ldots,c_n-b_n;\ c_1,\ldots,c_n;\right.$$

$$\frac{x_1}{1-x_{M+1}-\ldots-x_n},\ldots,\ \frac{x_M}{1-x_{M+1}-\ldots-x_n},\ \frac{-x_{M+1}}{1-x_{M+1}-\ldots-x_n},\ldots,$$

$$\left.\frac{-x_n}{1-x_{M+1}-\ldots-x_n}\right).$$

4. TRANSFORMTION OF THE CONFLUENT MULTIPLE HYPERGEOMETRIC FUNCTION ${}^{(k)}_{(2)}\phi^{(n)}_{AC}$.

Since we can write

$$ {}^{(k)}_{(2)}\phi^{(n)}_{AC}\left(a,b_{k+1},\ldots,b_n;c_1,\ldots,c_n;\ x_1,\ldots,x_n\right)$$

$$= \frac{\Gamma(c_1)\ldots\Gamma(c_n)}{\Gamma(b_{k+1})\ldots\Gamma(b_n)}\left[1-x_1E_1-\ldots-x_nE_n\right]^{-a}\left\{\frac{\Gamma(b_{k+1})\ldots\Gamma(b_n)}{\Gamma(c_1)\ldots\Gamma(c_n)}\right\}$$

$$= \frac{\Gamma(c_1)\ldots\Gamma(c_n)}{\Gamma(b_{k+1})\ldots\Gamma(b_n)}\left[1-x_{m+1}-\ldots-x_n\right]^{-a}$$

$$\left[1-\frac{x_1E_1+\ldots+x_mE_m+x_{m+1}\Delta_{m+1}+\ldots+x_n\Delta_n}{1-x_{m+1}-\ldots-x_n}\right]^{-a}\left\{\frac{\Gamma(b_{k+1})\ldots\Gamma(b_n)}{\Gamma(c_1)\ldots\Gamma(c_n)}\right\}$$

Therefore, we derive

(4.1) $${}^{(k)}_{(2)}\phi^{(n)}_{AC}\left(a,b_{k+1},\ldots,b_n;\ c_1,\ldots,c_n;\ x_1,\ldots,x_n\right)$$

$$= \left(1-x_{m+1}-\ldots-x_n\right)^{-a}\ {}^{(k)}_{(2)}\phi^{(n)}_{AC}\left(a,b_{k+1},\ldots,b_m,c_{m+1}-b_{m+1},\ldots,c_n-b_n;c_1,\ldots,c_n;\right.$$

$$\frac{x_1}{1 - x_{m+1} - \ldots - x_n}, \ldots, \frac{x_m}{1 - x_{m+1} - \ldots - x_n}, \frac{-x_{m+1}}{1 - x_{m+1} - \ldots - x_n}, \ldots,$$

$$\left. \frac{-x_n}{1 - x_{m+1} - \ldots - x_n} \right),$$

which for $m = k$ gives (2.6); while for $k = 0$ and $m = k$, it reduces to (3.2).

Thus, by specializing the parameters and variables, we can derive different transformation formulas for hypergeometric functions of one, two, three and four variables. These transformation formulas may be useful in other topics of mathematical analysis.

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