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ON METRIC DENSITY AS A SET FUNCTION

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ABSTRACT : This paper endeavours to reveal the topological character of the subclasses $\mathcal{D}(x)$ and $\mathcal{M}(x)$ (in the class of all Lebesgue measurable subsets of $\mathbb{R}$) defined by Martin in his paper [3] and shows that in this respect, they are similar to their duals $\mathcal{D}(E)$ and $\mathcal{D}^*(E)$ which are the collection of all points at which E has a density and that subcollection of it where this value of density is either 'zero' or 'one'.

Key words : Metric (Lebesgue) density, Borel, $F_{\sigma\delta}$, $\overline{D}_x$ -measurable, finitely additive outer measure.

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1. INTRODUCTION

Given a Lebesgue measurable set E ($\subseteq\mathbb{R}$) and x ($\in\mathbb{R}$), quantities such as $\overline{D}_x(E) = \limsup\limits_{I\to x}\dfrac{\mu(E\cap I)}{\mu(I)} = \sup\limits_{\{I_K\}}\left\{\limsup\limits_{k\to\infty}\dfrac{\mu(E\cap I_k)}{\mu(I_k)};I_k\to x\right\}$ and $\underline{D}_x(E) = \liminf\limits_{I\to x}\dfrac{\mu(E\cap I)}{\mu(I)} = \inf\limits_{\{I_k\}}\left\{\liminf\limits_{k\to\infty}\dfrac{\mu(E\cap I_k)}{\mu(I_k)};I_k\to x\right\}$ (where each I_k is a non-degenerate interval of $\mathbb{R}$ and $I_k \to x$ means that x ($\in I_k$) for each k ($\in\mathbb{N}$) and $\mu(I_k)\to 0$, μ being the Lebesgue measure in $\mathbb{R}$) are called respectively as the upper and lower metric (or Lebesgue) densities of E at x. In particular if, $\overline{D}_x(E) = \underline{D}_x(E)$, E is said to possess a density at the point x which is the common value of the above two quantitites and is denoted by $D_x(E)$.

By virtue of the Density theorem of Lebesgue, given any linear measurable set E,

$$\begin{aligned} D_x(E) &= 1, \quad \mu\text{ - almost everywhere on } E \\ &= 0, \quad \mu\text{ - almost everywhere on } \mathbb{R}\backslash E. \end{aligned}$$

Thus the concept of density as defined by Lebesgue and put to use in his famous theorem is usually considered as a point function where the set is kept fixed and the point is made to vary.

Thus given any linear measurable set E, as per the density theorem stated above, we can define a point function

$$\overline{D}_E : \mathbb{R} \to [0, 1] \text{ such that -}$$

$$\overline{D}_E(x) = D_x(E) = 1, \mu \text{ - almost everywhere on } E$$

$$= 0, \mu \text{ - almost everywhere on } \mathbb{R}\backslash E$$

which accordingly gives rise to the following two sets

$$\mathcal{D}(E) = \{x \in \mathbb{R} \mid D_x(E) \text{ exists}\}$$

and $\mathcal{D}^*(E) = \{x \in \mathbb{R} \mid D_x(E) = 0 \text{ or } 1\}$

But this concept of density may also be viewed in an entirely different context, where in place of the set, the point is fixed and a function.

$\overline{D}_x : \mathcal{M} \to [0, 1]$ is defined on the class $\mathcal{M}$ of all Lebesgue measurable sets of $\mathbb{R}$ by means of the formula–

$$\overline{D}_x(E) = \overline{D}_E(x) \text{ for every } E \ (\in \mathcal{M})$$

Martin [2] found that the set function $\overline{D}_x$ is a finitely sub-additive outer measure function on the class $\mathcal{M}$ of Lebesgue measurable sets and more interestingly the class $\mathcal{M}(x)$ of $\overline{D}_x$ - measurable sets can be given the following precise formulation–

$$\mathcal{M}(x) = \{E \in \mathcal{D}(x);\ D_x(E) = 0 \text{ or } 1\} \quad (*)$$

Now if $\mathcal{D}(x)$ stands for the class of all measurable sets for which $D_x(E)$ exists, then the set $\mathcal{D}(E)$ and the class $\mathcal{D}(x)$ can be considered as dual of each other (with the role of set and point interchanged) and a similar relation holds between $\mathcal{D}^*(E)$ and $\mathcal{M}(x)$. Moreover, viewed in this context, the assertion (*) may be regarded as the dual formulation of the density theorem stated above.

Here in this paper, we endeavour to unravel an interesting similarity in the topological character of the two structures $\mathcal{D}(E)$ and $\mathcal{D}(x)$ (and also of $\mathcal{D}^*(E)$ and $\mathcal{M}(x)$) which according to the above mode of representing their definitions are dual of each other.

It is already known that in the topology of the real line derived from the usual metric structure in it, the sets $\mathcal{D}(E)$ and $\mathcal{D}^*(E)$ are Borel $F_{\sigma\delta}$ sets. Results of this type follow as immediate consequences of more general theorems (related to interval functions) proved by the present author in [1]. There it was proved that for any Lebesgue measurable set E (of

$\mathbb{R}^n$), both the sets $\mathcal{D}(E)$ and $\mathcal{D}^*(E)$ are of the Borel $F_{\sigma\delta}$ type. It is interesting to note from the deductions given in the following paragraphs that their dual structures viz. $\mathcal{D}(x)$ and $\mathcal{M}(x)$ (with the role of set and point interchanged) also bear the same property of being Borel sets of the same Borel type in the topology of measurable subsets of the real line derived from the pseudo-metric $\sigma.\mathcal{M} \times \mathcal{M} \to \mathbb{R}$ defined by —

$$\sigma(E, F) = \mu(E \Delta F) \text{ where } E, F\ (\in \mathcal{M}).$$

Main Results. The results here are derived as particular cases of more general theorems concerning Lebesgue integrable functions on $\mathbb{R}$, the class being denoted by the symbol $\mathcal{L}$. For any x ($\in \mathbb{R}$) chosen and fixed we define—

$$\Phi_x(f) = \limsup_{I \to x} \frac{\int_I f d\mu}{\mu(I)} \qquad (f \in \mathcal{L})$$

$$\Psi_x(f) = \liminf_{I \to x} \frac{\int_I f d\mu}{\mu(I)}$$

which acording to their definition are both real-valued functions on $\mathcal{L}$. We also set,

$$A(x) = \left\{ f \in \mathcal{L}: \lim_{I \to x} \frac{\int_I f d\mu}{\mu(I)} \text{ exists} \right\},$$

$$\text{and } \mathcal{L}(x) = \left\{ f \in \mathcal{L}: \limsup_{I \to x} \frac{\int_I f d\mu}{\mu(I)} = 0 \text{ or } 1 \right\}$$

It is well-known that the class $\mathcal{L}$ is a metric space with respect to the pseudo-metric $\rho(f, g) = \int_R |f - g| d\mu$.

The following proposition illustrates two essential identities needed to proceed further and which are not hard to justify.

Proposition. If for any n ($\in \mathbb{N}$), $\mathcal{J}_x^{(n)} = \left\{ I;\ x \in I \text{ and } \frac{1}{n+1} < \mu(I) \le \frac{1}{n} \right\}$, then $\Phi_x = \limsup_{n \to \infty} \Phi_x^{(n)}$ and $\Psi_x = \liminf_{n \to \infty} \Psi_x^{(n)}$, where

$$\Phi_x^{(n)}(f) = \sup\left\{\frac{\int_I f d\mu}{\mu(I)}; I \in \mathcal{J}_x^{(n)}\right\} \text{ and}$$

$$\Psi_x^{(n)}(f) = \inf\left\{\frac{\int_I f d\mu}{\mu(I)}; I \in \mathcal{J}_x^{(n)}\right\}.$$

Theorem 1. $A(x)$ is a Borel $F_{\sigma\delta}$ subset of $(\mathcal{L}, \rho)$.

Lemma 1.1. For any n ($\in \mathbb{N}$), both $\Phi_x^{(n)}$ and $\Psi_x^{(n)}$ are continuous functions or $\mathcal{L}$.

Proof of the Lemma. For any f, g ($\in \mathcal{L}$).

$$\left|\Phi_x^{(n)}(f) - \Phi_x^{(n)}(g)\right| = \left|\sup\left\{\frac{\int_I f d\mu}{\mu(I)}; I \in \mathcal{J}_x^{(n)}\right\} - \sup\left\{\frac{\int_I g d\mu}{\mu(I)}; I \in \mathcal{J}_x^{(n)}\right\}\right|$$

$$\leqq \sup\left\{\frac{\int_I |f - g|}{\mu(I)}; I \in \mathcal{J}_x^{(n)}\right\}$$

$$\leqq n.\int_R |f - g| d\mu \ldots$$

and this inequality is enough to exhibit the continuity of $\Phi_x^{(n)}$.

Also, $$\left|\Psi_x^{(n)}(f) - \Psi_x^{(n)}(g)\right| = \left|\inf\left\{\frac{\int_I f d\mu}{\mu(I)}; I \in \mathcal{J}_x^{(n)}\right\} - \inf\left\{\frac{\int_I g d\mu}{\mu(I)}; I \in \mathcal{J}_x^{(n)}\right\}\right|$$

$$\leqq \sup\left\{\frac{\int_I |f - g|}{\mu(I)}; I \in \mathcal{J}_x^{(n)}\right\} \leqq n\int_R |f - g| d\mu$$

which similarly shows that the function $\Psi_x^{(n)}$ is continuous.

Lemma 1.2. For any r ($\in$ R), the sets $\{\Phi_x \geqq r\}$ and $\{\Psi_x \leqq r\}$ are G_δ in $(\mathcal{L}, \rho)$.

Proof. By virtue of the proposition stated in the beginning of this section,

$$\{f \in \mathcal{L};\ \Phi_x(f) \geqq r\} = \bigcap_p \bigcap_n \bigcup_{k \geqq n} \left\{\Phi_x^{(k)}(f) > r - \frac{1}{p}\right\}$$

and
$$\{f \in \mathcal{L};\ \Psi_x(f) \leqq r\} = \bigcap_p \bigcap_n \bigcup_{k \geqq n} \left\{\Psi_x^{(k)}(f) < r + \frac{1}{p}\right\}.$$

Now using Lemma 1.1, the result follows.

Proof of the theorem. We may write,

$$A_x = \mathcal{L} \backslash \{f \in \mathcal{L};\ \Psi_x(f) < \Phi_x(f)\} = \mathcal{L} \backslash \bigcup_k \{f \in \mathcal{L};\ \Psi_x(f) < r_k < \Phi_x(f)\}$$

where $\{r_k;\ k \in \mathbb{N}\}$ is the set of all rationals in $\mathbb{R}$. But for any k, the set

$$\{f \in \mathcal{L};\ \Psi_x(f) < r_k < \Phi_x(f)\} = \{f \in \mathcal{L};\ \Psi_x(f) < r_k\} \cap \{f \in \mathcal{L};\ \Phi_x(f) > r_k\}$$

$$= \bigcup_q \bigcup_q \left[\left\{f \in \mathcal{L}; \Psi_x(f) \leqq r_k - \frac{1}{p}\right\} \cap \left\{f \in \mathcal{L}; \Phi_x(f) \geqq k + \frac{1}{q}\right\}\right]$$

is $G_{\delta\sigma}$ by Lemma 1.2.

Hence the theorem.

For any n ($\in \mathbb{N}$), we denote by the symbol $J_x^{(n)}$ the interval $\left(x - \frac{1}{n}, x + \frac{1}{n}\right)$ and write

$$\eta_x(f) = \lim_{I \to x} \frac{\int_I f d\mu}{\mu(I)} \quad \text{and} \quad \eta_x^{(n)}(f) = \lim_{I \to x} \frac{\int_{J_x^{(n)}} f d\mu}{\mu\left(J_x^{(n)}\right)}.$$

Theorem 2. $\mathcal{L}(x)$ is a Borel $F_{\sigma\delta}$ subset of $\mathcal{L}$.

Lemma 2.1. The function $\eta_x^{(n)}$ is continuous on $A(x)$.

Proof. Let f, g ($\in A(x)$). Then

$$\left|\eta_x^{(n)}(f)-\eta_x^{(n)}(g)\right| = \left|\frac{\int_{J_x^{(n)}} f d\mu}{\mu\left(J_x^{(n)}\right)} - \frac{\int_{J_x^{(n)}} g d\mu}{\mu\left(J_x^{(n)}\right)}\right|$$

$$= \frac{n}{2}\left|\int_{J_x^{(n)}} f d\mu - \int_{J_x^{(n)}} g d\mu\right| \leqq \frac{n}{2}\int_R |f-g| d\mu$$

which finally settles the lemma.

Lemma 2.2. For any r ($\in \mathbb{R}$), the set $\{f \in \mathcal{L}, \eta_x(f) = r\}$ is $F_{\sigma\delta}$ in $A(x)$.

Proof.

$$\{f \in A(x);\ \eta_x(f) = r\} =$$

$$\bigcap_q \left[\left\{f \in A(x); \eta_x(f) < r + \frac{1}{q}\right\} \cap \left\{f \in A(x); \eta_x(f) > r - \frac{1}{q}\right\}\right]$$

But $\eta_x = \lim_{n\to\infty} \eta_x^{(n)}$ and therefore,

$$\left\{f \in A(x); \eta_x(f) > r - \frac{1}{q}\right\} = \bigcup_k \bigcup_p \zeta_{k,p}^{(q)}(x), \text{ where}$$

$$\zeta_{k,p}^{(q)}(x) = \bigcap_{n \geqq p} U_{k,n}^{(q)}(x) \text{ and } U_{k,n}^{(q)}(x) = \left\{f \in A(x);\ \eta_x^{(n)}(f) \geqq r - \frac{1}{q} + \frac{1}{k}\right\}$$

So by lemma 2.1., $\left\{f \in A(x); \eta_x(f) > r - \frac{1}{q}\right\}$ and similarly $\left\{f \in A(x); \eta_x(f) < r + \frac{1}{q}\right\}$ are F_σ and therefore the above set is an $F_{\sigma\delta}$ subset of $A(x)$.

Proof of the theorem. Since $\mathcal{L}(x) = \{f \in \mathcal{L};\ \eta_x(f) = 0 \text{ or } 1\}$ and $A(x)$ being an $F_{\sigma\delta}$ subset of $\mathcal{L}$ (by theorem 1), the theorem follows.

Thus we have proved in terms of Theorem 1 and 2 that for any fixed x in $\mathbb{R}$, the subclasses $A(x)$ and $\mathcal{L}(x)$ of the class $\mathcal{L}$ of Lebesgue integrable functions on $\mathbb{R}$ are Borel $F_{\sigma\delta}$ subsets of it. Had we chose instead of $\mathcal{L}$, the class $\mathcal{M}$ of Lebesgue measurable subsets of $\mathbb{R}$ to start with (which means that subclass of $\mathcal{L}$ consisting precisely of the characteristics functions of measurable sets) we would have reached as a consequence the following two conclusions–

Corollary 1. $\mathcal{D}(x)$ is a Borel $F_{\alpha\delta}$ subset of $(\mathcal{M}, \sigma)$

and

Corollary 2. $\mathcal{M}(x)$ is a Borel $F_{\sigma\delta}$ subset of (M, σ).

This shows that just as the sets $\mathcal{D}(E)$ and $\mathcal{D}^*(E)$ (for a fixed Lebesgue measurable set E) are Borel subsets of $\mathbb{R}$ (with its usual topology), so are their duals $\mathcal{D}(x)$ and $\mathcal{M}(x)$ (when the roles of the set and point are interchanged) having the same Borel type, in the topology of Lebesgue measurable subsets of $\mathbb{R}$.

Note. Using translation invariance of Lebesgue measure, it may be easily seen that the metric spaces $(\mathcal{D}(x), \sigma)$ and $(\mathcal{D}(y), \sigma)$ are isometric and therefore homeomorphic. The same reason suffices to show that $\mathcal{M}(x)$ and $\mathcal{M}(y)$ are also homeomorphic. Now if we write $\mathcal{M}_0(x) = \{E \in \mathcal{D}(x) : D_x(E) = 0\}$ and $\mathcal{M}_1(x) = \{E \in \mathcal{D}(x) : D_x(E) = 1\}$, then also $\mathcal{M}_0(x)$ and $\mathcal{M}_0(y)$ are isometric and therefore homeomorphic as Lebesgue measure is translation invariant and so also are $\mathcal{M}_0(x)$ and $\mathcal{M}_1(y)$ because of this fact and also that $\sigma(E\Delta F) = \sigma((\mathbb{R}\setminus E)\Delta(\mathbb{R}\setminus F))$ for any two $E, F(\in \mathcal{M})$. But $\mathcal{D}(x)$ and $\mathcal{M}(y)$ are not homeomorphic as it can be shown by virtue of a more general deduction [2] that both $\mathcal{M}_0(x)$ and $\mathcal{M}_1(x)$ are dense in $(\mathcal{M}, \sigma)$.

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