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MINIMAL STRUCTURES, *M*-CONTINUITY FOR MULTIFUNCTIONS AND BITOPOLOGICAL SPACES

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ABSTRACT : By using *M*-continuity between *m*-spaces, we establish the unified theory of several modified forms of continuity for multifunctions between bitopological spaces.

Key words and phrases : *m*-structure, *m*-open (*i, j*)-*M*-continuous, *M*-continuous, bitopological space, multifunction.

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1. INTRODUCTION

Semi-open sets, preopen sets, α-open sets and β-open sets play an important role in the researches of generalizations of continuity in topological spaces and bitopological spaces. By using these sets, many authors introduced and studied various types of modifications of continuity for functions and multifunctions in topological spaces and bitopological spaces. The notion of irresolute functions was introduced b Crossley and Hildebrand [8]. This notion was extended to multifunctions by Ewert and Lipski [11] and studied in [40] and [43]. The notion of preirresolute multifunctions is introduced in [41] and studied in [25]. The notion of α-irresolute multifunctions is introduced by Noiri and Nasef [34] and studied in [44]. Recently, Abd El-Monsef and Nasef [3] introduced and studied the notion of γ-irresolute multifunctions. Furthermore, β-irresolute multifunctions are studied in [40] and almost irresolute multifunctions are studied in [48].

Maheshwari and Prasad [23] and Bose [6] introduced the concepts of semi-open sets and semi-continuity in bitopological spaces. Jelić [12], Kar and Bhattacharyya [14] and Khedr et al. [16] introduced the concepts of preopen sets and precontinuity in bitopological spaces. The notions of α-open sets (or feebly open sets) and α-continuity (or feeble continuity) in bitopological spaces were studied in [13], [31] and [18]. The notions of semi-preopen sets and semi-precontinuity in bitopologiçal spaces were studied in [16]. The notion of irresolute functions in bitopological spaces is introduced and studied in [24]. In [32] Nasef studied almost quasi-continuous functions in bitopological spaces. Some generaliztions of irresolute functions in bitopological spaces are studied in [18], [31] and [17].

Recently, the present authors [45], [46] introduced and investigated the notions of minimal structures, m-spaces, m-continuity and M-continuity for functions. Moreover, in [36], we extended the concept of M-continuity to functions in bitoopological spaces. In the present paper, we introduce multifunctions between bitopological spaces called (i, j)- upper/lower M-continuous. The multifunctions turn out generalizations of the following functions and multifunctions:

(1) some irresolute multifunctions in topological spaces,

(2) modifications of continuous functions in bitopological spaces,

(3) (i, j)-M-continuous functions in bitopological spaces.

We present some characterizations and properties of (i, j)-upper/lower M-continuous multifunctions in bitopological spaces. In the last section, we introduce several new types of (i, j)-upper/lower M-continuous multifunctions in bitopological spaces.

2. PRELIMINARIES

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be *regular closed* (resp. *regular open*) if $\mathrm{Cl}(\mathrm{Int}(A)) = A$(resp. $\mathrm{Int}(\mathrm{Cl}(A)) = A$). A subset A is said to be *δ-open* [50] if for each $x \in A$ there exists a regular open set G such that $x \in G \subset A$. A point $x \in X$ is called a *δ-cluster* point of A if $\mathrm{Int}(\mathrm{Cl}(V) \cap A \neq \emptyset$ for every open set V containing x. The set of all δ-cluster points of A is called the *δ-closure* of A and is denoted by $\mathrm{Cl}_\delta(A)$. The set $\{x \in X : x \in U \subset A$ for some regular open set U of $X\}$ is called the *δ-interior* of A and is denoted by $\mathrm{Int}_\delta(A)$.

Definition 2.1. Let (X, τ) be a topological space. A subset A of X is said to be *α-open* [33] (resp. *semi-open* [21], *preopen* [27], *β-open* [1], *b-open* [4] or *γ-open* [9]) if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$).

The family of all semi-open (resp. preopen, α-open, β-open, γ-open) sets in X is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\gamma(X)$).

Definition 2.2. [The complement of a semi-open (resp. preopen, α-open, β-open, γ-open) set is said to be *semi-closed* [7] (resp. *preclosed* [10], *α-closed* [28], *β-closed* [1], *γ-closed* [9]).

Definition 2.3. The intersection of all semi-closed (resp. preclosed, α-closed, β-closed, γ-closed) sets of X containing A is called the *semi-closure* [7] (resp. *preclosure* [10], *α-closure* [28], *β-closure* [2], *γ-closure* [9] of A and is denoted by $\mathrm{sCl}(A)$ (resp. $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, ${}_\beta\mathrm{Cl}(A)$, ${}_\gamma\mathrm{Cl}(A)$).

Definition 2.4. The union of all semi-open (resp. preopen, α-open, β-open, γ-open) sets of X contained in A is called the *semi-interior* (resp. *preinterior*, *α-interior*, *β-interior*, *γ-interior*) of A and is denoted by $\mathrm{sInt}(A)$ (resp. $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, ${}_\beta\mathrm{Int}(A)$, ${}_\gamma\mathrm{Int}(A)$).

Throughout the present paper (X, τ) and (Y, σ) always denote topological spaces and (X, τ_1, τ_2) and (Y, σ_1, σ_2) denote bitopological spaces. The closure of A and the interior of A with respect to τ_i are denoted by $i\mathrm{Cl}(A)$ and $i\mathrm{Int}(A)$, respectively. A point $x \in X$ is called an *(i, j)-θ-closure point* [15] (resp. *(i, j)-δ-contact point* [5]) of a subset A if $j\mathrm{Cl}(U) \cap A \neq \emptyset$ (resp. $i\mathrm{Int}(j\mathrm{Cl}(U)) \cap A \neq \emptyset$) for every τ_i-open set U of X containing x. The set of all (i, j)-θ-closure (resp. (i, j)-δ-contact) points of A is denoted by $(i, j)\mathrm{Cl}_\theta(A)$ (resp. $(i, j)\mathrm{Cl}_\delta(A)$). If $A = (i, j)\mathrm{Cl}_\theta(A)$ (resp. $A = (i, j)\mathrm{Cl}_\delta(A)$), then A is called *(i, j)-θ-closed* (resp. *(i, j)-δ-closed*). The complement of an (i, j)-θ-closed (resp. (i, j)-δ-closed) set is said to be *(i, j)-θ-open* (resp. *(i, j)-δ-open*). It is proved in [15] that the family of all (i, j)-θ-open (resp. (i, j)-δ-open) sets, denoted by $(i, j)\theta(X)$ (resp. $(i, j)\delta(X)$), is a topology for X. A subset A of a bitopological space (X, τ_1, τ_2) is (i, j)-regular open if $A = i\mathrm{Int}(j\mathrm{Cl}(A))$.

For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a subset B of a space Y by $F^+(B)$ and $F^-(B)$, respectively, that is

$$F^+(B) = \{x \in X : F(x) \subset B\} \text{ and } F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

Definition 2.5. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be

a) *upper-irresolute* [11] (resp. *upper-preirresolute* [41], *upper-α-irresolute* [34], *upper-β-irresolute* [42], *upper-γ-irresolute* [3], *upper almost irresolute* [48]) if $F^+(V)$ is semi-open (resp. preopen, α-open, β-open, γ-open, β-open) for each semi-open (resp. preopen, α-open, β-open, γ-open, semi-open) set V of (Y, σ),

(b) *lower-irresolute* [11] (resp. *lower-preirresolute* [41], *lower-α-irresolute* [34], *lower-β-irresolute* [42], *lower-γ-irresolute* [3], *lower almost irresolute* [48]) if $F^-(V)$ is semi-open (resp. preopen, α-open, β-open, τ-open, β-open) for each semi-open (resp. preopen, α-open, β-open, γ-open, semi-open) set V of (Y, σ).

3. MINIMAL STRUCTURES AND *M*-CONTINUITY

Definition 3.1. A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (or briefly *m-structure*) [45], [46] on X if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X) (or briefly (X, m)), we denote a nonempty set X with a minimal structure m_X on X and call it an *m-space*. Each member of m_X is said to be *m_X-open* (or briefly *m-open*) and the complement of an m_X-open set is said to be *m_X-closed* (or briefly *m-closed*).

Remark 3.1. Let (X, τ) be a topological space. Then the families $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$ and $\gamma(X)$ are all m-structures on X.

Definition 3.2. Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the *m_X-closure* of A and the *m_X-interior* of A are defined in [26] as follows:

(1) $m\mathrm{Cl}(A) = \cap\{F : A \subset F,\ X - F \in m_X\}$,

(2) $m\mathrm{Int}(A) = \cup\{U : U \subset A,\ U \in m_X\}$.

Remark 3.2. Let (X, τ) be a topological space and A be a subset of X. If $m_X = \tau$ (resp. $SO(X)$, $PO(X)$, $\alpha(X)$, $\beta(X)$, $\gamma(X)$), then we have

(a) $m\mathrm{Cl}(A) = \mathrm{Cl}(A)$ (resp. $s\mathrm{Cl}(A)$, $p\mathrm{Cl}(A)$, $\alpha\mathrm{Cl}(A$, ${}_{\beta}\mathrm{Cl}(A)$, ${}_{\gamma}\mathrm{Cl}(A)$),

(b) $m\mathrm{Int}(A) = \mathrm{Int}(A)$ (resp. $s\mathrm{Int}(A)$, $p\mathrm{Int}(A)$, $\alpha\mathrm{Int}(A)$, ${}_{\beta}\mathrm{Int}(A)$, ${}_{\gamma}\mathrm{Int}(A)$).

Lemma 3.1. (Maki et al. [26]). *Let (X, m_X) be an m-space. For subsets A and B of X, the following properties hold:*

(1) $m\mathrm{Cl}(X - A) = X - m\mathrm{Int}(A)$ *and* $m\mathrm{Int}(X - A) = X - m\mathrm{Cl}(A)$,

(2) *If* $(X - A) \in m_X$, *then* $m\mathrm{Cl}(A) = A$ *and if* $A \in m_X$, *then* $m\mathrm{Int}(A) = A$,

(3) $m\mathrm{Cl}(\emptyset) = \emptyset$, $m\mathrm{Cl}(X) = X$, $m\mathrm{Int}(\emptyset) = \emptyset$ *and* $m\mathrm{Int}(X) = X$,

(4) *If* $A \subset B$, *then* $m\mathrm{Cl}(A) \subset m\mathrm{Cl}(B)$ *and* $m\mathrm{Int}(A) \subset m\mathrm{Int}(B)$,

(5) $A \subset m\mathrm{Cl}(A)$ *and* $m\mathrm{Int}(A) \subset A$,

(6) $m\mathrm{Cl}(m\mathrm{Cl}(A)) = m\mathrm{Cl}(A)$ *and* $m\mathrm{Int}(A)) = m\mathrm{Int}(A)$.

Lemma 3.2. (Popa and Noiri [45]). *Let (X, m_X) be an m-space and A a subset of X. Then $x \in m\mathrm{Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing x.*

Definition 3.3. A minimal structure m_X on a nonempty set X is said to have *property* $\mathcal{B}$ [26] if the union of any family of subsets belonging to m_X belongs to m_X.

Lemma 3.3. (Popa and Noiri [47]). *Let (X, m_X) be an m-space and m_X have property $\mathcal{B}$. Then for a subset A of X, the following properties hold:*

(1) $A \in m_X$ *if and only if* $m\mathrm{Int}(A) = A$,

(2) *A is m-closed if and only if* $m\mathrm{Cl}(A) = A$,

(3) $m\mathrm{Int}(A) \in m_X$ *and* $m\mathrm{Cl}(A)$ *is m-closed.*

Definition 3.4. A multifunction $F : (X,s\ m_X) \to (Y, m_Y)$ is said to be

(1) *upper M-continuous* [35] if for each $x \in X$ and each $V \in m_Y$ containing $F(x)$, there exists $U \in m_X$ containing x such that $F(U) \subset V$,

(2) *lower M-continuous* [35] if for each $x \in X$ and each $V \in m_Y$ such that $F(x) \cap V \neq \emptyset$, there exists $U \in m_X$ containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$.

Theorem 3.1. (Noiri and Popa [35] *For a multifunction $F : (X, m_X) \to (Y, m_Y)$, where m_Y has property $\mathcal{B}$, the following properties are equivalent:*

(1) *F is upper M-continuous;*

(2) $F^+(V) = m\mathrm{Int}(F^+(V))$ *for every* $V \in m_Y$;

(3) $F^-(K) = m\mathrm{Cl}(F^-(K))$ *for every* m_Y*-closed set* K *of* Y;

(4) $m\mathrm{Cl}(F^-(B)) \subset F^-(m\mathrm{Cl}(B))$ *for every subset* B *of* Y;

(5) $F^+(m\mathrm{Int}(B)) \subset m\mathrm{Int}(F^+(B))$ *for every subset* B *of* Y.

Corollary 3.1. *Let* m_X *and* m_Y *be m-structures with property* $\mathcal{B}$. *For a multifunction* F: $(X, m_X) \to (Y, m_Y)$, *the following properties are equivalent:*

(1) *F is upper M-continuous;*

(2) $F^+(V)$ *is* m_X*-open for every* V m_Y;

(3) $F^-(K)$ *is* m_X*-closed for every* m_Y*-closed set* K *of* Y.

Proof. The proof follows from Theorem 3.1 and Lemma 3.3.

Theorem 3.2. (Noiri and Popa [35]) *For a multifunction* $F : (X, m_X) \to (Y, m_Y)$, *where* m_Y *has property* $\mathcal{B}$, *the following properties are equivalent:*

(1) *F is lower M-continuous;*

(2) $F^-(V) = m\mathrm{Int}(F^-(V))$ *for every* $V \in m_Y$;

(3) $F^+(K) = m\mathrm{Cl}(F^+(K))$ *for every* m_Y*-closed set* K *of* Y;

(4) $m\mathrm{Cl}(F^+(B)) \subset F^+(m\mathrm{Cl}(B))$ *for every subset* B *of* Y;

(5) $F^-(m\mathrm{Int}(B)) \subset m\mathrm{Int}(F^-(B))$ *for every subset* B *of* Y;

(6) $F(m\mathrm{Cl}(A)) \subset m\mathrm{Cl}(F(A))$ *for every subset* A *of* X.

Corollary 3.2. *Let* m_X *and* m_Y *be m-structures with property* $\mathcal{B}$. *For a multifunction* F : $(X, m_X) \to (Y, m_Y)$, *the following properties are equivalent:*

(1) *F is lower M-continuous;*

(2) $F^-(V)$ *is* m_X*-open for every* $V \in m_Y$;

(3) $F^+(K)$ *is* m_X*-closed for every* m_Y*-closed set* K *of* Y.

Proof. The proof follows from Theorem 3.2 and Lemma 3.3.

4. MINIMAL STRUCTURES AND BITOPOLOGICAL SPACES

First, we recall some definitions of weak forms of open sets in a bitopological space.

Definition 4.1. A subset A of a bitopological space (X, τ_1, τ_2) is said to be

(1) *(i, j)-semi-open* [23] if $A \subset j\mathrm{Cl}(i\mathrm{Int}(A))$, where $i \neq j$, $i, j = 1, 2$,

(2) *(i, j)-preopen* [12] if $A \subset i\mathrm{Int}(j\mathrm{Cl}(A))$, where $i \neq j$, $i, j = 1, 2$,

(3) *(i, j)-α-open* [13] if $A \subset i\mathrm{Int}(j\mathrm{Cl}(i\mathrm{Int}(A)))$, where $i \neq j$, $i, j = 1, 2$,

(4) *(i, j)-semi-preopen* [16] if there exists an (i, j)-preopen set U such that $U \subset A \subset j\mathrm{Cl}(U)$, where $i \neq j$, $i, j = 1, 2$,

(5) *faintly semi-open* [29] if $A \subset \tau_2\text{-}\mathrm{Cl}(\tau_1\text{-}\mathrm{Int}(A)) \cup \tau_1\text{-}\mathrm{Cl}(\tau_2\text{-}\mathrm{Int}(A))$.

The family of (i, j)-semi-open (resp. (i, j)-preopen, (i, j)-α-open, (i, j)-semi-preopen) sets of (X, τ_1, τ_2) is denoted by $(i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{SPO}(X)$).

Remark 4.1. Let (X, τ_1, τ_2) be a bitopological space and A a subset of X. Then $(i, j)\mathrm{SO}(X)$, $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$ and $(i, j)\mathrm{SPO}(X)$ are all m-structures on X. Hence, if $m_X^{ij} = (i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{SPO}(X)$), then we have

(1) $m_X^{ij}\mathrm{Cl}(A) = (i, j)\text{-}\mathrm{sCl}(A)$ [23] (resp. $(i, j)\text{-}\mathrm{pCl}(A)$ [16], $(i, j)\text{-}\alpha\mathrm{Cl}(A)$ [31], $(i, j)\text{-}\mathrm{spCl}(A)$ [16]),

(2) $m_X^{ij}\mathrm{Int}(A) = (i, j)\text{-}\mathrm{sInt}(A)$ (resp. $(i, j)\text{-}\mathrm{pInt}(A)$, $(i, j)\text{-}\alpha\mathrm{Int}(A)$, $(i, j)\text{-}\mathrm{spInt}(A)$).

Remark 4.2. Let (X, τ_1, τ_2) be a bitopological space. Then

(1) Let $m_X^{ij} = (i, j)\mathrm{SO}(X)$ (resp. $(i, j)\alpha(X)$). Then by Lemma 3.1 we obtain the results established in Theorem 13 of [23] (resp. Theorem 3.6 of [31]).

(2) Let $m_X^{ij} = (i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{SPO}(X)$). Then, by Lemma 3.2 we obtain the results established in Theorem 1.15 of [22] (resp. Theorem 3.5 of [16], Theorem 3.5 of [31], Theorem 3.5 of [16]).

Remark 4.3. Let (X, τ_1, τ_2) be a bitopological space. Then

(1) $(i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{SPO}(X)$) is an m-structure on X satisfying property $\mathcal{B}$ by Theorem 2 of [23] (resp. Theorem 4.2 of [14] or Theorem 3.2 of [16], Theorem 3.2 of [31], Theorem 3.2 of [16]).

(2) Let $m_X^{ij} = (i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{SPO}(X)$). Then, by Lemma 3.3 we obtain the results established in Theorem 1.13 of [22] (resp. Theorem 3.5 of [16], Theorem 3.6 of [31], Theorem 3.6 of [16]).

5. *M*-CONTINUITY AND BITOPOLOGICAL SPACES

Definition 5.1. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i, j)-irresolute* [24] (resp. *(i, j)-preirresolute* [18], *(i, j)-α-irresolute* or *(i, j)-feebly continuous* [31], *(i, j)-β-irresolute* [42], *(i, j)-δ-continuous* [17], *FS-irresolute* [29] if $f^{-1}(V)$ is (i, j)-semi-open (resp. (i, j)-preopen, (i, j)-α-open, (i, b)-β-open, (i, j)-δ-open, faintly semi-open) in X for each (i, j)-semi-open (resp. (i, j)-preopen, (i, j)-α-open, (i, j)-β-open, (i, j)-δ-open, faintly semi-open) set V of Y.

Definition 5.2. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i, j)-faintly semi-continuous* [19] (resp. *(i, j)-faintly precontinuous* [19], *(i, j)-faintly β-continuous* [19] if $f^{-1}(V)$ is (i, j)-semi-open (resp. (i, j)-preopen, (i, j)-β-open) in X for each (i, j)-θ-open set V of Y.

Definition 5.3. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i, j)-almost quasi-continuous* [30] if $f^{-1}(V)$ is (i, j)-semi-open in X for each (i, j)-regular-open set V of Y.

Definition 5.4. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be

(1) *(i, j)-upper δ-continuous* [20] if $F^{+}(V) \in (i, j)\delta(X)$ for each $V \in (i, j)\delta(Y)$,

(2) *(i, j)-lower δ-continuous* [20] if $F^{-}(V) \in (i, j)\delta(X)$ for each $V \in (i, j)\delta(Y)$.

Remark 5.1. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces. Let m_X^{ij} (resp. m_Y^{ij}) be an m-structure on X (resp. Y) determined by τ_1 and τ_2 (resp. σ_1 and σ_2). In case $m_X^{ij} = (i, j)\delta(X)$ and $m_Y^{ij} = (i, j)\delta(Y)$, a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (i, j)-upper/lower δ-continuous if and only if a multifunction $F : (X, m_X^{ij}) \to (Y, m_Y^{ij})$ is upper/lower M-continuous.

Now, we can state the main definition of the present paper as follows:

Definition 5.5. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces. Let m_X^{ij} (resp. m_Y^{ij}) be an m-structure on X (resp. Y) determined by τ_1 and τ_2 (resp. σ_1 and σ_2). A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i, j)-upper/lower M-continuous* if a multifunction $F : (X, m_X^{ij}) \to (Y, m_Y^{ij})$ is upper/lower M-continuous.

Remark 5.2. (1) The (i, j)-upper/lower δ-continuous multifunction is a particular case of (i, j)-upper/lower M-continuous multifunction.

(2) The multifunction in Definition 5.5 is a generalization of each of the following functions:

(a) multifunctions defined by Definition 2.5,

(b) functions defined by Definitions 5.1, 5.2 and 5.3,

(c) functions defined by Definition 4.3 of [36].

By Definition 5.5, Theorems 3.1 and 3.2 and Corollaries 3.1 and 3.2, we obtain the following two theorems.

Theorem 5.1. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces. Let m^{ij}_X (resp. m^{ij}_Y) be an m-structure on X (resp. Y) determined by τ_1 and τ_2 (resp. σ_1 and σ_2), where m^{ij}_Y has property $\mathcal{B}$. Then for a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is (i, j)-upper M-continuous;*

(2) $F^+(V) = m^{ij}_X\text{-Int}(F^+(V))$ *for every* $V \in m^{ij}_Y$;

(3) $F^-(K) = m^{ij}_X\text{-Cl}(F^-(K))$ *for every m^{ij}_Y-closed set K of Y;*

(4) $m^{ij}_X\text{-Cl}(F^-(B)) \subset F^-(m^{ij}_Y\text{-Cl}(B))$ *for every sbset B of Y;*

(5) $F^+(m^{ij}_Y\text{-Int}(B)) \subset m^{ij}_X\text{-Int}(F^+(B))$ *for every subset B of Y.*

Corollary 5.1. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces. Let m^{ij}_X (resp. m^{ij}_Y) be an m-structure on X (resp. Y) determined by τ_1 and τ_2 (resp. σ_1 and σ_2), where m^{ij}_X and m^{ij}_Y have property $\mathcal{B}$. Then, for a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is (i, j)-upper M-continuous;*

(2) $F^+(V)$ *is m^{ij}_X-open for every m^{ij}_Y-open set V of Y;*

(3) $F^-(K)$ *is m^{ij}_X-closed for every m^{ij}_Y-closed set K of Y.*

Proof. This is an immediate consequence of Theorem 5.1 and Corollary 3.1.

Remark 5.3. If $m^{ij}_X = (i, j)\delta(X)$ and $m^{ij}_Y = (i, j)\delta(Y)$, then by Theorem 5.1 and Corollary 5.1 we obtain the results established in Theorem 2.3 of [20].

Theorem 5.2. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces. Let m^{ij}_X (resp. m^{ij}_Y) be an m-structure on X (resp. Y) determined by τ_1 and τ_2 (resp. σ_1 and σ_2), where m^{ij}_Y has property $\mathcal{B}$. Then, for a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is (i, j)-lower M-continuous;*

(2) $F^-(V) = m^{ij}_X\text{-Int}(F^-(V))$ *for every m^{ij}_Y-open set V of Y;*

(3) $F^+(K) = m^{ij}_X\mathrm{Cl}(F^+(K))$ *for every* m^{ij}_Y*-closed set* K *of* Y;

(4) $m^{ij}_X\mathrm{Cl}(F^+(B)) \subset F^+(m^{ij}_Y\mathrm{Cl}(B))$ *for every subset* B *of* Y;

(5) $F^-(m^{ij}_Y\mathrm{Int}(B)) \subset m^{ij}_X\mathrm{Int}(F^-(B))$ *for every subset* B *of* Y;

(6) $F(m^{ij}_X\mathrm{Cl}(A)) \subset m^{ij}_Y\mathrm{Cl}(F(A))$ *for every subset* A *of* X.

Corollary 5.2. *Let* (X, τ_1, τ_2) *and* (Y, σ_1, σ_2) *be bitopological spaces. Let* m^{ij}_X *(resp.* m^{ij}_Y*) be an m-structure on X (resp. Y) determined by* τ_1 *and* τ_2 *(resp.* σ_1 *and* σ_2*), where* m^{ij}_X *and* m^{ij}_Y *have property* $\mathcal{B}$. *Then, for a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, *the following properties are equivalent:*

(1) *F is (i, j)-lower M-continuous;*

(2) $F^-(V)$ *is* m^{ij}_X*-open for every* m^{ij}_Y*-open set* V *of* Y;

(3) $F^+(K)$ *is* m^{ij}_X*-closed for every* m^{ij}_Y*-closed set* K *of* Y.

Remark 5.4. (1) If $m^{ij}_X = (i, j)\delta(X)$ and $m^{ij}_Y = (i, j)\delta(Y)$, then by Theorem 5.2 and Corollary 5.2 we obtain the results established in Theorem 2.2 fo [20].

(2) In case $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, by Theorem 5.1 (or Theorem 5.2) and Corollary 5.1 (or Corollary 5.2), we obtain the results established in Theorem 4.1 and Corollary 4.1 of [36]).

6. SOME PROPERTIES OF (i, j)-UPPER/LOWER M-CONTINUITY

Definition 6.1. For a multifunction $F : (X, m_X) \to (Y, m_Y)$, we define a multifunction $\mathrm{mCl}(F)$: $(X, m_X) \to (Y, m_Y)$ as follows: $(\mathrm{mCl}(F))(x) = \mathrm{mCl}(F(x))$ for each $x \in X$.

Lemma 6.1. *If* $F : (X, m_X) \to (Y, m_Y)$ *is a multifunction, then* $(\mathrm{mCl}(F))^-(V) = F^-(V)$ *for each* m_Y*-open set* V *of* Y.

Proof. Let V be any m_Y-open set and $x \in (\mathrm{mCl}(F))^-(V)$. Then $V \cap (\mathrm{mCl}(F))(x) = V \cap \mathrm{mCl}(F(x)) \neq \emptyset$. Therefore, there exists $y \in V \cap \mathrm{mCl}(F(x))$. Since $y \in \mathrm{mCl}(F(x))$ and $y \in V \in m_Y$, by Lemma 3.2 $V \cap F(x) \neq \emptyset$ and hence $x \in F^-(V)$. Conversely, let $V \in m_Y$ and $x \in F^-(V)$, then $\emptyset \neq F(x) \cap V \subset \mathrm{mCl}(F(x)) \cap V = (\mathrm{mCl}(F))(x) \cap V$ and hence $x \in (\mathrm{mCl}(F))^-(V)$.

Theorem 6.1. *A multifunction* $F : (X, m_X) \rightarrow (Y, m_Y)$ *is lower M-continuous if and only if* mCl(F) *is lower M-continuous.*

Proof. *Necessity.* Suppose that F is lower M-continuous. Let $x \in X$ and V be any m_Y-open set of Y such that $(\mathrm{mCl}(F))(x) \cap V \neq \emptyset$. By Lemma 6.1 we have x $(\mathrm{mCl}(F))^-(V) = F^-(V)$ $= F^-(V)$ and hence $F(x) \cap V \neq \emptyset$. Since F is lower M-continuous, there exists $U \in m_X$ containing x such that $F(u) \cap V \neq \emptyset$ for eacy $u \in U$. Hence we have $(\mathrm{mCl}(F))(u) \cap V \neq \emptyset$ for each $u \in U$. This shows that mCl(F) is lower M-continuous.

Sufficiency. Suppose that mCl(F) is lower M-continuous. Let $x \in X$ and V be any m_Y-open set of Y such that $F(x) \cap V \neq \emptyset$. Then, by Lemma 6.1 we have $x \in F^-(V) = (\mathrm{mCl}(F))^-(V)$ and hence $(\mathrm{mCl}(F))(x) \cap V \neq \emptyset$. Since mCl($F$) is lower M-continuous, there exists $U \in m_X$ containing x such that $(\mathrm{mCl}(F))(u) \cap V \neq \emptyset$ for each $u \in U$. By Lemma 6.1, we have $u \in (\mathrm{mCl}(F))^-(V) = F^-(V)$ for each $u \in U$. Thus, we have $F(u) \cap V \neq \emptyset$ for each $u \in U$. Thus, F is lower M-continuous.

Remark 6.1. Let (X, τ) and (Y, σ) be topological spaces. if m_X = SO(X) (resp. PO(X), $\alpha(X)$, $\beta(X)$, $\gamma(X)$ and m_Y = SO(Y) (resp. PO(Y), $\alpha(Y)$, $\beta(Y)$, $\gamma(Y)$), then by Theorem 6.1 we obtain the result established in Theorem 2 of [43] (resp. theorem 3.5 of [41] and Theorem 6 of [25], Theorem 6 of [44], Theorem 3.6 of [40], Theorem 3.5 of [3]).

Corollary 6.1. *Let* (X, τ_1, τ_2) *and* (Y, σ_1, σ_2) *be bitopological spaces. Let* m_X^{ij} *(resp.* m_Y^{ij}*) be an m-structure on X (resp. Y) determined by* τ_1 *and* τ_2 *(resp.* σ_1 *and* σ_2*). Then, a multifunction* $F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ *is (i, j)-lower M-continuous if and only if mCl(F)* : (X, τ_1, τ_2) $\rightarrow (Y, \sigma_1, \sigma_2)$ *is (i, j)-lower M-continuous.*

Proof. This follows from Definition 5.5 and Theorem 6.1.

Definition 6.2. An m-space (X, m_X) is said to be *m-compact* [35] if every cover of X by m_X-open sets has a finite subcover. A subset K of (X, m_X) is said to be *m-compact* [44] if every cover of K by m_X-open sets has a finite subcover.

Theorem 6.2. (Noiri and Popa [35]) *Let* (Y, m_Y) *be an m-space and* m_Y *an m-structure with property* $\mathcal{B}$. *If* $F : (X, m_X) \rightarrow (Y, m_Y)$ *is an upper M-continuous multifunction such that F(x) is m-compact for each* $x \in X$ *and K is an m-compact set of X, then F(K) is m-compact.*

Definition 6.3. Let (X, τ_1, τ_2) be a bitopological space and m_X^{ij} an m-structure determined by τ_1 and τ_2. a subset K of X is said to be *(i, j)-m-compact* if K is m_X^{ij}-compact.

Corollary 6.2. *Let* (X, τ_1, τ_2) *and* (Y, σ_1, σ_2) *be bitopological spaces and* m_X^{ij} *(resp.* m_Y^{ij}*)*

an m-structure on X (resp. Y) determined by τ_1 *and* τ_2 *(resp.* σ_1 *and* σ_2*), where* m_Y^{ij} *has property* $\mathcal{B}$. *If* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is an (i, j)-upper M-continuous multifunction such that* $F(x)$ *is (i, j)-m-compact for each* $x \in X$ *and K is an (i, j)-m-compact set of X, then F(K) is (i, j)-m-compact in Y.*

Proof. This is an immediate consequence of Definitions 5.5 and 6.2 and Thorem 6.2.

Remark 6.2. If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a function, then by Corollary 6.2 we obtain the result established in Theorem 5.2 of [36].

Definition 6.4. A multifunction $F : (X, m_X) \to (Y, m_Y)$ is said to be

(1) *upper M-continuous* at a point $x \in X$ if for each $V\ m_Y$ containing $F(x)$, there exists $U \in m_X$ containing x such that $F(U) \subset V$,

(2) *lower M-continuous* at a point $x \in X$ if for each $V \in m_Y$ such that $F(x) \cap V \neq \emptyset$, there exists $U \in m_X$ containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$.

Definition 6.5. Let (X, m_X) be an m-space and A a subset of X. The m_X*-frontier* of A, denoted by $\mathrm{mFr}(A)$ [46], is defined by $\mathrm{mFr}(A) = \mathrm{mCl}(A) \cap \mathrm{mCl}(X - A) = \mathrm{mCl}(A) - \mathrm{mInt}(A)$.

Theorem 6.3. *The set of all points* $x \in X$ *at which a multifunction* $F : (X, m_X) \to (Y, m_Y)$ *is not upper/lower M-continuous is identical with the union of the* m_X*-frontiers of upper/lower inverse images of* m_Y*-open sets containing/meeting F(x).*

Proof. Let x be a point of (X, m_X) at which F is not upper M-continuous. Then, there exists $V \in m_Y$ containing $F(x)$ such that $U \cap (X - F^+(V)) \neq \emptyset$ for every $U \in m_X$ containing x. By Lemma 3.2, we have $x \in \mathrm{mCl}(X - F^+(V))$. Since $x \in F^+(V)$, we have $x \in \mathrm{mCl}(F^+(V))$ and hence $x \in \mathrm{mFr}(F^+(V))$. Conversely, let $V \in m_Y$ containing $F(x)$ and $x \in \mathrm{mFr}(F^+(V))$. Now, assume that F is upper M-continuous at x, then there exists $U \in m_X$ containing x such that $F(U) \subset V$; hence $U \subset F^+(V)$. Therefore, we obtain $x \in \mathrm{mInt}(F^+(V))$. This is a contradiction. Therefore, F is not upper M-continuous. Since the proof for lower M-continuous multifunctions is similar, it is omitted.

Corollary 6.3. *Let* (X, τ_1, τ_2) *and* (Y, σ_1, σ_2) *be bitopological spaces and* m_X^{ij} *(reesp.* m_Y^{ij}*) an m-structure on X (resp. Y) determined by* τ_1 *and* τ_2 *(resp.* σ_1 *and* σ_2*), where* m_Y^{ij} *has property* $\mathcal{B}$. *The set of all points* $x \in X$ *which a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is not (i, j)-upper/lower M-continuous is identical with the union of the* m_X^{ij}*-frontiers of upper/lower inverse images of* m_Y^{ij}*-open sets containing/meeting F(x).*

Remark 6.3. If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a function, then by Corollary 6.3 we obtain the result established in Theorem 5.4 of [36].

7. NEW FORMS OF (i, j)-M-CONTINUOUS MULTIFUNCTIONS

There are many modification sof open sets in topological spaces. First, we recall θ-closed sets due to Velicko [50]. Let (X, τ) be a topological space and A a subset of X. A point $x \in X$ is a θ-cluster point of A if $\mathrm{Cl}(V) \cap A \neq \emptyset$ for evry open set V containing x. The set of all θ-cluster points of A is called the *θ-closure* of A and is denoted by $\mathrm{Cl}_\theta(A)$. If $A = \mathrm{Cl}_\theta(A)$, then A is said to be *θ-closed* [50]. The complement of a θ-closed set is said to be *θ-open*. The union of all θ-open sets contained in A is called the *θ-interior* of A and is denoted by $\mathrm{Int}_\theta(A)$.

Let (X, τ_1, τ_2) be a bitopological space and A a subset of X. The δ-closure (resp. θ-closure) of A and the δ-interior (resp. θ-interior) of A with respect to τ_i are denoted by ${}_i\mathrm{Cl}_\delta(A)$ (resp. ${}_i\mathrm{Cl}_\theta(A)$) and $i\mathrm{Int}_\delta(A)$ (resp. $i\mathrm{Int}_\theta(A)$). The notions of δ-semiopen sets [39] and δ-preopen sets [49] are generalized in [37] and [38] to the setting of bitopological spaces as follows:

Definition 7.1. A subset A of a bitopological space (X, τ_1, τ_2) is said to be

(1) *(i, j)-δ-semi-open* [37] if $A \subset j\mathrm{Cl}(i\mathrm{Int}_\delta(A))$, where $i \neq j$, $j = 1, 2$,

(2) *(i, j)-δ-preopen* [38] if $A \subset i\mathrm{Int}(j\mathrm{Cl}_\delta(A))$, where $i \neq j$, $i, j = 1, 2$,

(3) *(i, j)-δ-b-open* if $A \subset i\mathrm{Int}(j\mathrm{Cl}_\delta(A)) \cup j\mathrm{Cl}(i\mathrm{Int}_\delta(A))$, where $i \neq j$, $i, j = 1, 2$,

(4) *(i, j)-δ-semipreopen* (simply *(i, j)-δ-sp-open*) if there exists an (i, j)-δ-preopen set U such that $U \subset A \subset j\mathrm{Cl}(U)$, where $i \neq j\mathrm{Cl}(U)$, where $i \neq j$, $i, j = 1, 2$.

Definition 7.2. A subset A of a bitopological space (X, τ_1, τ_2) is said to be

(1) *(i, j)-θ-semi-open* if $A \subset j\mathrm{Cl}(i\mathrm{Int}_\theta(A))$, where $i \neq j$, $i, j = 1, 2$,

(2) *(i, j)-θ-preopen* if $A \subset i\mathrm{Int}\ (j\mathrm{Cl}_\theta(A))$, where $i \neq j$, $i, j = 1, 2$,

(3) *(i, j)-θ-b-open* if $A \subset i\mathrm{Int}(j\mathrm{Cl}_\theta(A)) \cup j\mathrm{Cl}(i\mathrm{Int}_\theta(A))$, where $i \neq j$, $i, j = 1, 2$,

(4) *(i, j)-θ-semipreopen* (simply *(i, j)-θ-sp-open*) if there exists an (i, j)-θ-preopen set U such that $U \subset A \subset j\mathrm{Cl}(U)$, where $i \neq j$, $i, j = 1, 2$.

Let (X, τ_1, τ_2) be a bitopological space. The family of (i, j)-δ-semi-open (resp. (i, j)-δ-preopen, (i, j)-δ-b-open, (i, j)-δ-sp-open, (i, j)-θ-semi-open, (i, j)-θ-preopen, (i, j)-θ-b-open, (i, j)-θ-sp-open) sets of (X, τ_1, τ_2) is denoted by $(i, j)\delta\mathrm{SO}(X)$ (resp. $(i, j)\delta\mathrm{PO}(X)$, $(i, j)\delta\mathrm{BO}(X)$, $(i, j)\delta\mathrm{SPO}(X)$, $(i, j)\theta\mathrm{SO}(X)$, $(i, j)\theta\mathrm{PO}(X)$, $(i, j)\theta\mathrm{BO}(X)$, $(i, j)\theta\mathrm{SPO}(X)$).

Remark 7.1. Let (X, τ_1, τ_2) be a bitopological space. The families $(i, j)\delta\mathrm{SO}(X)$, $(i, j)\delta\mathrm{PO}(X)$, $(i, j)\delta\mathrm{BO}(X)$, $(i, j)\delta\mathrm{SPO}(X)$, $(i, j)\theta\mathrm{SO}(X)$, $(i, j)\theta\mathrm{PO}(X)$, $(i, j)\theta\mathrm{BO}(X)$ and $(i, j)\theta\mathrm{SPO}(X)$ are all m-structurs with property $\mathcal{B}$.

For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ we can define many new types of (i, j)-upper/lower M-continuous multifunctions. For example, in case $m_X^{ij} = (i, j)\delta SO(X)$ (resp. $(i, j)\delta PO(X)$, $(i, j)\delta BO(X)$, $(i, j)\delta SPO(X)$, $(i, j)\theta SO(X)$, $(i, j)\theta PO(X)$, $(i, j)\theta BO(X)$, $(i, j)\theta SPO(X)$ and $m_Y^{ij} = (i, j)\delta SO(Y)$ (resp. $(i, j)\delta PO(Y)$, $(i, j)\delta BO(Y)$, $(i, j)\delta SPO(Y)$, $(i, j)\theta SO(Y)$, $(i, j)\theta PO(Y)$, $(i, j)\theta BO(Y)$, $(i, j)\theta SPO(Y)$), we can define new types of (i, j)-upper/lower M-continuous multifunctions as follows:

Definition 7.3. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be

(1) *(i, j)-upper/lower δ-semi-irresolute* if $F : (X, (i, j)\delta SO(X)) \to (Y, (i, j)\delta SO(Y))$ is upper/lower M-continuous,

(2) *(i, j)-upper/lower δ-preirresolute* if $F : (X, (i, j)\delta PO(X)) \to (Y, (i, j)\delta PO(Y))$ is upper/lower M-continuous,

(3) *(i, j)-upper/lower δ-b-irresolute* if $F : (X, (i, j)\delta BO(X)) \to (Y, (i, j)\delta BO(Y))$ is upper/lower M-continuous,

(4) *(i, j)-upper/lower δ-sp-irresolute* if $F : (X, (i, j)\delta SPO(X)) \to (Y, (i, j)\delta SPO(Y))$ is upper/lower M-continuous.

Definition 7.4. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be

(1) *(i, j)-upper/lower θ-semi-irresolute* if $F : (X, (i, j)\theta SO(X)) \to (Y, (i, j)\theta SO(Y))$ is upper/lower M-continuous,

(2) *(i, j)-upper/lower θ-preirresolute* if $F : (X, (i, j)\theta PO(X)) \to (Y, (i, j)\theta PO(Y))$ is upper/lower M-continuous,

(3) *(i, j)-upper/lower θ-b-irresolute* if $F : (X, (i, j)\theta BO(X)) \to (Y, (i, j)\theta BO(Y))$ is upper/lower M-continuous,

(4) *(i, j)-upper/lower θ-sp-irresolute* if $F : (X, (i, j)\theta SPO(X)) \to (Y, (i, j)\theta SPO(Y))$ is upper/lower M-continuous.

Conclusion. We can apply the results established in Sections 5 and 6 for the following multifunctions:

(1) the multifunctions defined in Definitions 7.3 and 7.4 and

(2) any (i, j)-upper/lower M-continuous multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ defined by using any m-structures m_X^{ij} and m_Y^{ij}.

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