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# ON A GENERALIZATION OF HILBERT'S INEQUALITY

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**ABSTRACT :** In this paper, by introducing a parameter $\lambda$, we give a generalization of Hilbert's inequality, we also consider its equivalent form and the corresponding integral form.

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## 1. INTRODUCTION

If $0<\sum_{n=1}^{\infty} a_n^2<\infty$, and $0<\sum_{n=1}^{\infty} b_n^2<\infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}<\pi\left\{\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}b_n^2\right\}^{1/2}, \qquad \dots (1.1)$$

where the constant $\pi$ is best possible (see Hardy et. al. [1]). (1.1) is well known as Hilbert's inequality, which is important in analysis and applications (see Mitrinovic [2]). Its equivalent form is

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^2<\pi^2\sum_{n=1}^{\infty}a_n^2, \qquad \dots (1.2)$$

where the constant $\pi^2$ is best possible. The corresponding integral form of (1.1) and (1.2) are :

If $0<\int_0^{\infty} f^2(t)dt<\infty$, and $0<\int_0^{\infty} g^2(t)dt<\infty$, then

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy<\pi\left\{\int_0^{\infty} f^2(x)dx\int_0^{\infty} g^2(x)dx\right\}^{1/2}; \qquad \dots (1.3)$$

$$\int_0^{\infty}\left(\int_0^{\infty}\frac{f(x)}{x+y}dx\right)^2 dy<\pi^2\int_0^{\infty}f^2(x)dx, \qquad \cdots (1.4)$$

where the constant $\pi$ and $\pi^2$ are still best possible.

In recent years, Gao [3,4] and Pachpatte [5] made some improvements of (1.1). By introducing a paramater $\lambda$, Yang [6] gave a generalization of (1.3) as

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{(x+y)^{\lambda}}dxdy$$

$$<B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left\{\int_0^{\infty}x^{1-\lambda}f^2(x)dx\int_0^{\infty}x^{1-\lambda}g^2(x)dx\right\}^{1/2}, \ (0<\lambda<1). \qquad \cdots (1.5)$$

where the constant $B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$ is $\beta$-function $B(p,q)$ for $p = q = \lambda/2$.

Recently, Kuang [7] proved the following inequality

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x^{\lambda}+y^{\lambda}}dxdy$$

$$<\frac{\pi}{\lambda\sin(\pi/2\lambda)}\left\{\int_0^{\infty}x^{1-\lambda}f^2(x)dx\int_0^{\infty}x^{1-\lambda}g^2(x)dx\right\}^{1/2}, \ \left(\frac{1}{2}<\lambda<1\right) \qquad \cdots (1.6)$$

for $\lambda = 1$, both (1.5) and (1.6) reduce to (1.3).

In this paper we use methods similar to those of Gao and Yang [4] and Kuang [8] to generalize (1.1) with a best possible constant factor. We also consider its equivalent form which is more general than (1.6).

## 2. THE EXTENDED SERIES FORM

**Lemma 2.1.** *For $\lambda > 0$, define the weight function $\omega_{\lambda}(y)$ and the weight coefficient as $\tilde{\omega}_{\lambda}(n)$ respectively by*

$$\omega_{\lambda}=\int_0^{\infty}(x^{\lambda}+y^{\lambda})^{-1}\left(\frac{y^{\lambda}}{x^{\lambda}}\right)^{\frac{1}{\lambda}-\frac{1}{2}}dx, \ y\in(0,\infty), \qquad \cdots (2.1)$$

$$\tilde{\omega}_\lambda = \sum_{m=1}^{\infty} \frac{1}{m^\lambda + n^\lambda}\left(\frac{n^\lambda}{m^\lambda}\right)^{\frac{1}{\lambda}-\frac{1}{2}} n \in N. \qquad \ldots (2.2)$$

*Then we have* $\omega_\lambda(y) = y^{1-\lambda}\pi/\lambda$, $y \in (0, \infty)$, *and for* $0 < \lambda \le 2$,

$$\tilde{\omega}_\lambda(n) < \omega_\lambda(n), \ n \in N.$$

**Proof.** Substituting $t = {x^\lambda}/{y^\lambda}$, we find

$$\omega_\lambda(y) = \frac{1}{\lambda}\int_0^\infty \frac{1}{y^\lambda(1+t)}\left(\frac{1}{t}\right)^{\frac{1}{\lambda}-\frac{1}{2}} y t^{\frac{1}{\lambda}-1} dt$$

$$= \frac{1}{\lambda} y^{1-\lambda}\int_0^\infty \frac{1}{(1+t)}\left(\frac{1}{t}\right)^{\frac{1}{2}} dt = \frac{\pi}{\lambda} y^{1-\lambda}.$$

Since, for $0 < \lambda \le 2$, $\frac{1}{\lambda}-\frac{1}{2} \ge 0$, then $f(x) = \frac{1}{x^\lambda + n^\lambda}\left(\frac{n^\lambda}{x^\lambda}\right)^{\frac{1}{\lambda}-\frac{1}{2}}$ is strictly decreasing on $(0,\infty)$. Hence, we have $\tilde{\omega}_\lambda(n) < \omega_\lambda(n)$, $n \in N$. The lemma is proved. ∎

**Lemma 2.2.** *For* $\lambda > 0$ *and* $0 < \varepsilon < \lambda/2$, *we have*

$$\int_1^\infty \frac{1}{x^{1+\varepsilon}}\int_0^{1/x^\lambda} \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{2}+\frac{\varepsilon}{2\lambda}} du dx = O(1)\left(\varepsilon \to 0^+\right). \qquad \ldots (2.3)$$

**Proof.** Since $0 < \varepsilon < \lambda/2$, then we have

$$0 < \int_1^\infty \frac{1}{x^{1+\varepsilon}}\int_0^{1/x^\lambda} \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{2}+\frac{\varepsilon}{2\lambda}} du dx$$

$$< \int_1^\infty \frac{1}{x^{1+\varepsilon}}\int_0^{1/x^\lambda} \left(\frac{1}{u}\right)^{\frac{1}{2}+\frac{\varepsilon}{2\lambda}} du dx$$

$$< \int_1^\infty \frac{1}{x}\int_0^{1/x^\lambda} \left(\frac{1}{u}\right)^{\frac{3}{4}} du dx$$

$$= 4\int_1^\infty x^{-1-\lambda/4} dx = 16/\lambda.$$

This shows that (2.3) is valid. The lemma is proved. ∎

**Theorem 2.3.** *If* $0 < \lambda \le 2,\ 0<\sum_{n=1}^{\infty} n^{1-\lambda} a_n^2 <\infty$ *and* $0<\sum_{n=1}^{\infty} n^{1-\lambda} b_n^2 <\infty$ *then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m^\lambda + n^\lambda} < \frac{\pi}{\lambda}\left\{\sum_{n=1}^{\infty} n^{1-\lambda} a_n^2 \sum_{n=1}^{\infty} n^{1-\lambda} b_n^2\right\}^{1/2}, \quad \cdots (2.4)$$

*where the constant* $(\pi/\lambda)$ *is best possible. In particular, for* $\lambda = 1/2,\ 2,$ *we have*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{\sqrt{m}+\sqrt{n}} < 2\pi\left\{\sum_{n=1}^{\infty}\sqrt{n}a_n^2\sum_{n=1}^{\infty}\sqrt{n}b_n^2\right\}^{1/2}; \quad \cdots (2.5)$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m^2+n^2} < \frac{\pi}{2}\left\{\sum_{n=1}^{\infty}\frac{1}{n}a_n^2\sum_{n=1}^{\infty}\frac{1}{n}b_n^2\right\}^{1/2}, \quad \cdots (2.6)$$

*where the constants* $2\pi$ *and* $\pi/2$ *are best possible.*

**Proof.** By Cauchy's inequlity and Lemma 2.1, we obtain

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_m}{m^\lambda + n^\lambda}$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m}{(m^\lambda+n^\lambda)^{1/2}}\left(\frac{m^\lambda}{n^\lambda}\right)^{\frac{1}{2\lambda}-\frac{1}{4}}\frac{b_n}{(m^\lambda+n^\lambda)^{1/2}}\left(\frac{m^\lambda}{n^\lambda}\right)^{\frac{1}{2\lambda}-\frac{1}{4}}$$

$$\le \left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m^2}{(m^\lambda+n^\lambda}\left(\frac{m^\lambda}{n^\lambda}\right)^{\frac{1}{2\lambda}-\frac{1}{4}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{b_m^2}{(m^\lambda+n^\lambda}\left(\frac{n^\lambda}{m^\lambda}\right)^{\frac{1}{2\lambda}-\frac{1}{4}}\right\}^{1/2}$$

$$= \left\{\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}\frac{1}{(m^\lambda+n^\lambda)}\left(\frac{m^\lambda}{n^\lambda}\right)^{\frac{1}{2\lambda}-\frac{1}{4}}\right]a_m^2\right.$$

$$\left.\times\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{1}{(m^\lambda+n^\lambda)}\left(\frac{n^\lambda}{m^\lambda}\right)^{\frac{1}{2\lambda}-\frac{1}{4}}\right]b_n^2\right\}^{1/2}$$

$$= \left\{\sum_{m=1}^{\infty}\tilde{\omega}_{\lambda}(m)a_m^2\sum_{n=1}^{\infty}\tilde{\omega}_{\lambda}(n)b_n^2\right\}^{1/2}$$

$$< \left\{\sum_{m=1}^{\infty}\omega_{\lambda}(n)a_n^2\sum_{n=1}^{\infty}\omega_{\lambda}(n)b_n^2\right\}^{1/2}$$

$$= \frac{\pi}{\lambda}\left\{\sum_{n=1}^{\infty}n^{1-\lambda}a_n^2\sum_{n=1}^{\infty}n^{1-\lambda}b_n^2\right\}^{1/2}$$

Inequality (2.4) is thus proved.

For ε ∈ (0, λ/2), setting $\tilde{a}_n = \left(\frac{1}{n^{\lambda}}\right)^{\frac{1}{\lambda}-\frac{1}{2}+\frac{\varepsilon}{2\lambda}}$ (n N), then we have

$$\sum_{n=1}^{\infty}n^{1-\lambda}\tilde{a}_n^2 = \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} \text{ and}$$

$$\frac{1}{\varepsilon} = \int_1^{\infty}\frac{1}{x^{1+\varepsilon}}\,dx < \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} = 1+\sum_{n=2}^{\infty}\frac{1}{n^{1+\varepsilon}}$$

$$< 1+\int_1^{\infty}\frac{1}{x^{1+\varepsilon}}\,dx = 1+\frac{1}{\varepsilon}.$$

Hence, we find

$$\sum_{n=1}^{\infty}n^{1-\lambda}\tilde{a}_n^2 = \frac{1}{\varepsilon}+O(1) = \frac{1}{\varepsilon}(1+o(1)) \quad (\varepsilon \to 0^+); \qquad \ldots (2.7)$$

In view of (2.3), and $\int_0^{\infty}\frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{2}+\frac{\varepsilon}{2}}du = \pi+o(1)$ $(\varepsilon \to 0^+)$, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\tilde{a}_n\tilde{b}_n}{m^{\lambda}+n^{\lambda}} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^{\lambda}+n^{\lambda}}\left(\frac{1}{mn}\right)^{1-\frac{\lambda}{2}+\frac{\varepsilon}{2}}$$

$$> \int_1^{\infty}\left[\int_1^{\infty}\frac{1}{x^{\lambda}+y^{\lambda}}\left(\frac{1}{xy}\right)^{1-\frac{\lambda}{2}+\frac{\varepsilon}{2}}dy\right]dx$$

(Putting $u = y^{\lambda}/x^{\lambda}$)

$$= \frac{1}{\lambda}\int_1^{\infty}\frac{1}{x^{1+\varepsilon}}\int_{\frac{1}{x^\lambda}}^{\infty}\frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{2}+\frac{\varepsilon}{2\lambda}} dudx$$

$$= \frac{1}{\lambda}\left[\int_1^{\infty}\frac{1}{x^{1+\varepsilon}}\int_0^{\infty}\frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{2}+\frac{\varepsilon}{2\lambda}} dudx\right.$$

$$\left. - \left[\int_1^{\infty}\frac{1}{x^{1+\varepsilon}}\int_0^{\frac{1}{x^\lambda}}\frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{2}+\frac{\varepsilon}{2\lambda}} dudx\right]\right]$$

$$= \frac{1}{\lambda}\left[\frac{1}{\varepsilon}\int_0^{\infty}\frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{2}+\frac{\varepsilon}{2\lambda}} du - O(1)\right]$$

$$= \frac{1}{\lambda}\left[\frac{1}{\varepsilon}(\pi + o(1) - O(1))\right] = \frac{\pi}{\lambda\varepsilon}(1+o(1)) \quad (\varepsilon \to O^+) \qquad \dots (2.8)$$

Suppose there exists a positive integer K < $\pi/\lambda$, such that (2.6) is valid by changing $\pi/\lambda$ to $K$. Hence by (2.7) and (2.8), we get

$$\frac{\pi}{\lambda\varepsilon}(1+o(1)) < \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\tilde{a}_m\tilde{a}_n}{m^\lambda + n^\lambda} < K\sum_{n=1}^{\infty} n^{1-\lambda}\tilde{a}_n^2 = \frac{1}{\varepsilon}K(1+o(1)) \ (\varepsilon \to 0^+).$$

It follows that $\pi/\lambda \le K$, which contradicts the fact that K < $\pi/\lambda$. Hence the constant $\pi/\lambda$ in (2.6) is best possibel. The theorem is proved. ■

**Theorem 2.4.** *If* $0 < \lambda \le 2$, $0 < \sum_{n=1}^{\infty} n^{1-\lambda}a_n^2 < \infty$, *then*

$$\sum_{n=1}^{\infty} n^{\lambda-1}\left(\sum_{m=1}^{\infty}\frac{a_m}{m^\lambda + n^\lambda}\right)^2 < \left(\frac{\pi}{\lambda}\right)^2\sum_{n=1}^{\infty} n^{1-\lambda}a_n^2. \qquad \dots (2.9)$$

*Inequality (2.9) is equivalent to (2.6). The constant* $\left(\pi/\lambda\right)^2$ *in (2.9) is best possible. In particular, for* $\lambda$ = 1/2, 2, *we have*

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left(\sum_{m=1}^{\infty}\frac{a_m}{\sqrt{m}+\sqrt{n}}\right)^2 < 4\pi^2\sum_{n=1}^{\infty}\sqrt{n}\, a_n^2 ; \qquad \dots (2.10)$$

$$\sum_{n=1}^{\infty} n\left(\sum_{m=1}^{\infty}\frac{a_m}{m^2+n^2}\right)^2 < \frac{\pi^2}{4}\sum_{n=1}^{\infty}\frac{1}{n}a_n^2, \qquad \ldots (2.11)$$

*where the constants $4\pi^2$ and $\pi^2/4$ are best possible.*

**Proof.** Since $\sum_{n=1}^{\infty} n^{1-\lambda}a_n^2 > 0$, there exists $k_0 \in N$, such that for $k > k_0$, $\sum_{m=1}^{k}\frac{|a_m|}{m^\lambda+n^\lambda} > 0$.

Putting $b_n(k) = n^{1-\lambda}\sum_{m=1}^{k}\frac{|a_m|}{m^\lambda+n^\lambda}$ $(k > k_0)$, then, by (2.6), we have

$$\begin{aligned}0 < \sum_{n=1}^{k} n^{1-\lambda}b_n^2(k) &= \sum_{n=1}^{k} n^{\lambda-1}\left(\frac{|a_m|}{m^\lambda+n^\lambda}\right)^2 \\ &= \sum_{n=1}^{k}\sum_{m=1}^{k}\frac{|a_m|b_n(k)}{m^\lambda+n^\lambda} \\ &< \frac{\pi}{\lambda}\left\{\sum_{n=1}^{k} n^{1-\lambda}a_n^2\sum_{n=1}^{k} n^{1-\lambda}b_n^2(k)\right\}^{1/2},\end{aligned}$$

and then

$$\sum_{n=1}^{k} n^{1-\lambda}b_n^2(k) = \sum_{n=1}^{k} n^{\lambda-1}\left(\sum_{m=1}^{k}\frac{|a_m|}{m^\lambda+n^\lambda}\right)^2 < \left(\frac{\pi}{\lambda}\right)^2\sum_{n=1}^{k} n^{1-\lambda}a_n^2. \qquad \ldots (2.12)$$

Hence we find $0 < \sum_{n=1}^{\infty} n^{1-\lambda}b_n^2(\infty)$, and for , $k \to \infty$, by (2.6), we still have the inequality (2.12), and

$$\sum_{n=1}^{\infty} n^{\lambda-1}\left(\sum_{m=1}^{\infty}\frac{a_m}{m^\lambda+n^\lambda}\right)^2 \le \sum_{n=1}^{\infty} n^{\lambda-1}\left(\sum_{m=1}^{\infty}\frac{|a_m|}{m^\lambda+n^\lambda}\right)^2 < \left(\frac{\pi}{\lambda}\right)^2\sum_{n=1}^{\infty} n^{1-\lambda}a_n^2.$$

Inequality (2.9) is valid.

On the other hand, if (2.9) is valid, by Cauchy's inequality, we obtain

$$\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^\lambda+n^\lambda}\right)^2 = \left\{\sum_{n=1}^{\infty}\left[n^{(\lambda-1)/2}\sum_{m=1}^{\infty}\frac{a_m}{m^\lambda+n^\lambda}\right]\left[n^{(1-\lambda)/2}b_n\right]\right\}^2$$

$$\leq \sum_{n=1}^{\infty} n^{\lambda-1}\left(\sum_{m=1}^{\infty} \frac{a_m}{m^{\lambda}+n^{\lambda}}\right)^2 \sum_{n=1}^{\infty} n^{1-\lambda} b_n^2 . \qquad \cdots (2.13)$$

By (2.9), we have (2.6). Inequality (2.9) is equivalent to (2.6).

If the constant $\left(\pi/\lambda\right)^2$ in (2.9) is not best possible, using (2.13), we may get the same result that the constant $\pi/\lambda$ in (2.6) is not best possible, which is a contradiction. This proves the theorem. ∎

## 3. THE EXTENDED INTEGRAL FORM

**Theorem 3.1.** *If* $\lambda > 0,\ 0 < \int_0^{\infty} t^{1-\lambda} f^2(t)\, dt < \infty,$ *and* $0 < \int_0^{\infty} t^{1-\lambda} g^2(t)\, dt < \infty$ *then*

$$\int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{x^{\lambda}+y^{\lambda}}\, dxdy$$

$$< \frac{\pi}{\lambda}\left\{\int_0^{\infty} x^{1-\lambda} f^2(x)\, dx \int_0^{\infty} x^{1-\lambda} g^2(x)\, dx\right\}^{1/2}; \qquad \ldots (3.1)$$

$$\int_0^{\infty} y^{1-\lambda}\left(\int_0^{\infty} \frac{f(x)}{x^{\lambda}+y^{\lambda}}\, dx\right)^2 dy$$

$$< \left(\frac{\pi}{\lambda}\right)^2 \int_0^{\infty} x^{1-\lambda} f^2(x)\, dx, \qquad \ldots (3.2)$$

*where the constants* $\pi/\lambda$ *and* $\left(\pi/\lambda\right)^2$ *are best possible. Inequalities (3.1) and (3.2) are equivalent.*

**Proof.** By using Cauchy's ineqality, we obtain

$$\int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{x^{\lambda}+y^{\lambda}}\, dxdy$$

$$= \int_0^\infty \int_0^\infty \left[\frac{f(x)}{(x^\lambda + y^\lambda)^{1/2}}\left(\frac{x^\lambda}{y^\lambda}\right)^{\frac{1}{2\lambda}-\frac{1}{4}}\right]\left[\frac{f(x)}{(x^\lambda + y^\lambda)^{1/2}}\left(\frac{y^\lambda}{x^\lambda}\right)^{\frac{1}{2\lambda}-\frac{1}{4}}\right] dxdy$$

$$< \left\{\int_0^\infty \int_0^\infty \left[\frac{f^2(x)}{(x^\lambda + y^\lambda)^{1/2}}\left(\frac{x^\lambda}{y^\lambda}\right)^{\frac{1}{2\lambda}-\frac{1}{4}}\right] dydx\right.$$

$$\left.\times \int_0^\infty \int_0^\infty \left[\frac{g^2(x)}{(x^\lambda + y^\lambda)^{1/2}}\left(\frac{y^\lambda}{x^\lambda}\right)^{\frac{1}{2\lambda}-\frac{1}{4}}\right] dxdy\right\}^{1/2}. \qquad \ldots (3.3)$$

If (3.3) becomes eqality, then there exists numbers $A$ and $B$, such that (see Kuang [8], p. 29)

$$A\frac{f^2(x)}{(x^\lambda + y^\lambda)^{1/2}}\left(\frac{x^\lambda}{y^\lambda}\right)^{\frac{1}{2\lambda}-\frac{1}{4}} = B\frac{g^2(x)}{(x^\lambda + y^\lambda)^{1/2}}\left(\frac{y^\lambda}{x^\lambda}\right)^{\frac{1}{2\lambda}-\frac{1}{4}} \quad \text{a.e. in } (0, \infty) \times (0, \infty).$$

It follows that $x^{2-\lambda} f^2(x) = y^{2-\lambda} g^2(y)$ = const. a.e. in $(0, \infty) \times (0, \infty)$, which contradicts the fact that $0 < \int_0^\infty t^{1-\lambda} f^2(t)\, dt < \infty$. Hence (3.3) becomes a strict ineqality, and then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dxdy < \left\{\int_0^\infty \omega_\lambda(x) f^2(x)\, dx \int_0^\infty \omega_\lambda(y) g^2(y)\, dy\right\}^{1/2}.$$

By Lemma 2.1, we obtain (3.1).

If the constant $\pi/\lambda$ in (3.1) is not best possible, then there exists a positive number $K < \pi/\lambda$, such that (3.1) is valid by changing $(\pi/\lambda)$ to $K$. For $\varepsilon \in (\pi/\lambda)$, setting $\tilde{f}(t)$ as :

$\tilde{f}(t) = 0$, for $t \in (0,1)$ ; $\tilde{f}(t) = t^{-1+\frac{\lambda}{2}-\frac{\varepsilon}{2}}$, for $t \in [1, \infty)$, by (2.8), we have

$$\frac{\pi}{\lambda\varepsilon}(1 + o(1)) = \int_0^\infty \int_0^\infty \frac{\tilde{f}(x)\tilde{f}(y)}{x^\lambda + y^\lambda} dxdy$$

$$< k\int_0^\infty x^{1-\lambda}\tilde{f}^2(x)dx = \frac{k}{\varepsilon}(\varepsilon \to 0^+).$$

It follows that $\pi/\lambda \le k$, which contradicts the fact that $k < \pi/\lambda$. Hence the constant $\pi/\lambda$ in (3.1) is best possible.

There exists $T_0 > 0$, such that, for $T > T_0$, $\int_0^T \frac{|f(x)|}{(x^\lambda + y^\lambda)} dx > 0$. Writing

$$g(y,T) = y^{\lambda-1}\int_0^T \frac{|f(x)|}{(x^\lambda + y^\lambda)} dx, \ y \in (0,\infty),$$

we use (3.1) to obtain

$$\int_0^T y^{1-\lambda} g^2(y,T)\, dy = \int_0^T y^{\lambda-1}\left(\frac{|f(x)|}{x^\lambda + y^\lambda} dx\right) dy$$

$$= \int_0^T\int_0^T \frac{|f(x)|g(y,T)}{x^\lambda + y^\lambda}\, dxdy$$

$$< \frac{\pi}{\lambda}\left\{\int_0^T x^{1-\lambda} f^2(x)\, dx \int_0^T y^{1-\lambda} g^2(y,T)\, dy\right\}^{1/2} \quad \text{and}$$

$$\int_0^T y^{1-\lambda} g^2(y,T)\, dy = \int_0^T y^{1-\lambda}\left(\int_0^T \frac{|f(x)|}{x^\lambda + y^\lambda}\, dx\right)^2 dy$$

$$< \left(\frac{\pi}{\lambda}\right)^2 \int_0^T x^{1-\lambda} f^2(x)\, dx. \qquad \ldots (3.4)$$

Hence, we have $0 < \int_0^\infty y^{1-\lambda} g^2(y,\infty)\, dy < \infty$. For $T \to \infty$, still by (3.1), we have (3.4), and

$$\int_0^\infty y^{\lambda-1}\left(\int_0^\infty \frac{f(x)}{x^\lambda + y^\lambda} dx\right)^2 dy \le \int_0^\infty y^{\lambda-1}\left(\int_0^\infty \frac{|f(x)|}{x^\lambda + y^\lambda} dx\right)^2 dy < \left(\frac{\pi}{\lambda}\right)^2 \int_0^\infty x^{1-\lambda} f^2(x)\, dx.$$

Inequality (3.2) is valid. Following the result (2.12) in Teorem 2.3, we may show that (3.1) and (3.2) are equivalent. From the equivalence of (3.1) and (3.2), it follows that the constant $(\pi/\lambda)^2$ in (3.2) is best possible. The theorem is proved. ■

**Remarks 3.2.** *For $\lambda = 1$ (2.4) reduces to (1.1), so (2.4) is a new generalization of (1.1). Similarly, (2.5), (3.1) and (3.2) are new generalizations of (1.2), (1.3) and (1.4) respectively. Since the constant factor $\pi/\lambda$, in (3.1) for $\lambda \in (0, \infty)$, is best possible. Inequality (3.1) is more general than (1.6).*

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