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FURTHER GROWTH PROPERTIES OF COMPOSITE ENTIRE AND MEROMORPHIC FUNCTIONS

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ABSTRACT : In the paper we study the growth of composite entire and meromorphic functions which improve some known results.

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1. INTRODUCTION AND DEFINITIONS

Let f and g be two transcendental entire functions defined in the open complex plane C. It is well known [2] that $\lim_{r\to\infty}\frac{T(r, fog)}{T(r,f)} = \infty$ and $\lim_{r\to\infty}\frac{T(r, fog)}{T(r,g)} = \infty$. Singh [8] proved some comparative growth properties of log T(r, fog) and T(r, f). He [8] also raised the question of investigating the comparative growth of log T(r, fog) and T(r, g) which he was unable to solve. However, some results on the comparative growth of log T(r,fog) and T(r,g) are proved in [4]. In the paper we further investigate the above question of Singh [8] and study the comparative growth of log T(r, fog) with $T(r,f)\{\log T(r,f)\}^k$ and $T(r,g)\{\log T(r,g)\}^k$ where f is taken to be meromorphic, g is entire and k > 0.

If f and g are of positive lower order then Song and Yang [10] proved that

$$\lim_{r\to\infty}\frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r, f)} = \lim_{r\to\infty}\frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r, g)} = \infty, \text{ where}$$

$\log^{[k]} x = \log(\log^{[k-1]} x)$ for k = 1,2,3, ... and $\log^{[0]}x = x$.

Also in the sequel we use the following notation :

$\exp^{[k]}x = \exp(\exp^{[k-1]}x)$ for k = 1,2,3, ... and $\exp^{[0]}x = x$.

Since M(r, f) and M(r, g) are increasing function of r, Singh and Baloria [9] asked whether for sufficiently large R = R(r)

$$\limsup_{r\to\infty}\frac{\log^{[2]} M(r,\ fog)}{\log^{[2]} M(R, f)} < \infty \text{ and } \limsup_{r\to\infty}\frac{\log^{[2]} M(r,\ fog)}{\log^{[2]} M(R,\ g)} < \infty,$$

Singh and Baloria [9], Lahiri and Sharma [6], Liao and Yang [7] worked on this question. In the paper we throw some light on the comparative growth of $\log^{[2]}M(r, fog)$ and $\log M(r, g)$ where f and g are any two entire functions. We also study about the estimation of lower order of a composite meromorphic function whose left factor is of zero order. We do not explain the standard notations and definitions of the theory of entire and meromorphic functions as those are availabe in [11] and [3].

Definitions 1.1. The order ρ_f and lower order λ_f of a meromorphic function is defined as

$$\rho_f = \limsup_{r\to\infty}\frac{\log T(r,f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty}\frac{\log T(r,f)}{\log r}$$

If f is entire, one can easily verify that

$$\rho_f = \limsup_{r\to\infty}\frac{\log^{[2]} M(r,f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty}\frac{\log^{[2]} M(r,f)}{\log r}.$$

Definition 1.2. [7] Let f be a meromorphic function of order zero. Then ρ_f and λ_f are defined as follows :

$$\rho^*_f = \limsup_{r\to\infty}\frac{\log T(r,f)}{\log^{[2]} r} \text{ and } \lambda^*_f = \liminf_{r\to\infty}\frac{\log T(r,f)}{\log^{[2]} r}.$$

If f is entire, then clearly

$$\rho^*_f = \limsup_{r\to\infty}\frac{\log^{[3]} M(r,f)}{\log^{[2]} r} \text{ and } \lambda^*_f = \liminf_{r\to\infty}\frac{\log^{[3]} M(r,f)}{\log^{[2]} r}$$

Definition 1.3. The type σ_f of an entire function f is defined as

$$\rho_f = \limsup_{r\to\infty}\frac{\log M(r,f)}{r^{\rho_f}},\ 0 < \rho_f < \infty.$$

2. LEMMAS

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [2] If f and g are entire functions, for all sufficienctly large values or r,

$$M(r, fog) \geq M(\tfrac{1}{8}M(\tfrac{r}{2}, g) - |g(0)|, f).$$

Lemma 2.2. If f is meromorphic and g is entire then for all sufficiently large values of r,

Lemma 2.3. [5]. If f is a non constant entire function of finite order then

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{T(r,f)\{\log T(r,f)\}^k} = 0 \text{ where } k > 0.$$

Lemma 2.4. Let f be an entire function such that $0 < \rho_f < \infty$. If σ_f and $\sigma_f^{(k)}$ be the respective types of f and $f^{(k)}$ then $\sigma_f^{(k)} \le (2^k)^{\rho_f} \sigma_f$ where K = 0, 1, 2, 3, ...

Proof. It is known from G. Valiron {[11], p.35} that

$$\frac{1}{r}\{M(r,f) - |f(0)|\} \le M(r,f) \le \frac{1}{r} M(2r, f).$$

Noting that $\rho_f^{(k)} = \rho_f$ we get from the second part of the inequality for r ≥ 1,

$$M(r, f^{(k)}) \le M(2^k r, f)$$

or, $$\frac{\log M(r,f^{(k)})}{r^{\rho_{f^{(k)}}}} \le \frac{\log M(2^k r, f)}{(2^k r)^{\rho} f} . (2^k)^{\rho_f}$$

or, $$\limsup_{r\to\infty} \frac{\log M(r,f^{(k)})}{r^{\rho_{f^{(k)}}}} \le (2^k)^{\rho_f} \limsup_{r\to\infty} \frac{\log M(2^k r, f)}{(2^k r)^{\rho_f}}$$

or, $\sigma_f^{(k)} \le (2^k)^{\rho_f} \sigma_f$, which proves the lemma.

Lemma 2.5. Let f be meromorphic and g be entire such that $\lambda_g < \infty$. If $\lambda_{fog} = \infty$, then for every positive number A,

$$\lim_{r\to\infty} \frac{\log T(r, fog)}{\log^{[2]} M(r^A, g^{(k)})} = \infty \text{ where } k = 0, 1, 2, 3, \ldots\ldots .$$

Proof. Let us assume that the conclusion of the lemma does not hold.

Then there exists a constant B > 0 such that

$$\lim_{r\to\infty} \frac{\log T(r, fog)}{\log^{[2]} M(r^B, g^{(k)})} = \mu < \infty,$$ provided the limit exists. Then for all larger r,

$$\log T(r, fog) < (\mu + \varepsilon) \log^{[2]} M(r^B, g^{(k)}) \quad \ldots\ldots (1)$$

Again, for a sequence of values of r, tending to infinity,

$$\log^{[2]} M(r^B, g^{(k)}) < (\lambda_g(k)+\varepsilon)\, B \log r = (\lambda_g+\varepsilon)\, B \log r \qquad \ldots (2)$$

Thus combining (1) and (2) we get for a sequence of values of r tending to infinity

$\log T(r, fog) < (\mu + \varepsilon)(\lambda_g + \varepsilon)\, B \log r$

which implies that $\lambda_{fog} < \infty$. This is a contradiction.

Thus the lemma is proved.

3. THEOREMS

In this section we present the main results of the paper.

Theorem 3.1. Let f be meromorphic and g be non constant entire such that ρ_f and ρ_g are finite. Then

$$\liminf_{r\to\infty} \frac{\log T(r, fog)}{T(r, g)\{\log T(r, g)\}^k} = 0 \text{ where } k > 0.$$

Proof. By Lemma 2.2 and $T(r, g) \leq \log^+ M(r, g)$ we get for all sufficiently large values of r,

$$\log T(r,fog) \leq \log T(M(r,g), f) + \log\{1 + 0(1)\}$$

or, $$\log T(r,fog) < (\rho_f + \varepsilon) \log M(r,g) + \log\{1 + 0(1)\}$$

or, $$\frac{\log T(r, fog)}{T(r,g)\{\log T(r,g)\}^k} < \frac{(\rho_f+\varepsilon)\log M(r,g) + \log\{1+0(1)\}}{T(r,g)\{\log T(r,g)\}^k} \qquad \ldots (3)$$

Now by Lemma 2.3 it follows from (3),

$$\liminf_{r\to\infty} \frac{\log T(r, fog)}{T(r,g)\{\log T(r,g)\}} = 0.$$

This proves the theorem.

Remark 3.2. Considering f = g = exp z one can easily verify that no term in the denominator of $\dfrac{\log T(r, fog)}{T(r,g)\{\log T(r,g)\}^k}$ can be removed.

Remark 3.3. The condition $\rho_f < \infty$ in Theorem 3.1 is necessary which is evident from the following example.

Example 3.4. Let $f = \exp^{[2]} z$, $g = z$ and $k = 1$

Then and $\rho_f = \infty$ and $\rho_g = 0$.

Since $T(r, fog) \sim \dfrac{e^r}{(2\pi^3 r)^{1/2}}$ and

$T(r, g) \leq \log M(r, g) = \log r$ it follows that

$$\frac{\log T(r,fog)}{T(r,g)\{\log T(r,g)\}^k} \geq \frac{r - \frac{1}{2}\log r + O(1)}{\log r\{\log^{[2]} r\}}$$

which implies that

$$\lim_{r\to\infty} \frac{\log T(r,fog)}{T(r,g)\{\log T(r,g)\}^k} = \infty.$$

Theorem 3.5. Let f and g be two entire functions such that ρ_f and ρ_g are finite. Also let $\lambda_f > \rho_g$. Then $\liminf\limits_{r\to\infty} \dfrac{\log T(r,fog)}{T(r,f)\{\log T(r,f)\}^k} = 0$

Proof. Since $\lambda_g < \rho_f$ we can choose $\varepsilon(>0)$ in such a way that $\rho_g + \varepsilon < \lambda_f - \varepsilon$. By Lemma 2.2 and $T(r, g) \leq \log^+ M(r,g)$ we obtain fro all sufficiently large values of r,

$$\log T(r,fog) < (\rho_f + \varepsilon) \log M(r,g) + \log\{1 + O(1)\}$$

or, $$\frac{\log T(r,fog)}{T(r,f)\{\log T(r,f)\}^k}$$

$$< (\rho_f + \varepsilon)\frac{\log M(r,f)}{T(r,f)\{\log T(r,f)\}^k}\cdot\frac{\log M(r,g)}{\log M(r,f)} + \frac{\log\{1+o(1)\}}{T(r,f)\{\log T(r,f)\}^k} \qquad \ldots (4)$$

Again for all sufficiently large values of r,

$\log M(r,g) < r^{\rho_g + \varepsilon}$ and $\log M(r,f) > r^{\rho_g - \varepsilon}$.

Thus from (4) we obtain.

$$\frac{\log T(r,fog)}{T(r,f)\{\log T(r,f)\}^k}$$

$$\leq (\rho_f + \varepsilon) \frac{\log M(r,f)}{T(r,f)\{\log T(r,f)\}^k}\cdot\frac{r^{\rho_g+\varepsilon}}{r^{\lambda_f-\varepsilon}} + \frac{\log\{1+o(1)\}}{T(r,f)\{\log T(r,f)\}^k}$$

$$\text{or,}\quad \liminf_{r\to\infty} \frac{\log T(r,fog)}{T(r,f)\{\log T(r,f)\}^k}$$

$$\leq (\rho_f + \varepsilon) \liminf_{r\to\infty} \frac{\log M(r,f)}{T(r,f)\{\log T(r,f)\}^k} \cdot \lim_{r\to\infty} \frac{r^{\rho_g+\varepsilon}}{r^{\lambda_f-\varepsilon}} \qquad \ldots (5)$$

Now by Lemma 2.3, the theorem follows from (5).

Remarks 3.6. The following example shows that the condition $\lambda_f > \rho_g$ in Theorem 3.5 necessary.

Example 3.7. Let f = exp z, g = exp (z^2) and k = 1.

Then $\lambda_f = \rho_f = 1$ and $\rho_g = 2$.

Now 3T (2r,fog) ≥ log M(r,fog) = exp(r^2)

$$\text{or,}\quad \log T(r,fog) \geq \frac{r^2}{4} + O(1)$$

$$\text{and}\quad T(r,f) = \frac{r}{\pi}.$$

Thus it follows that

$$\frac{\log T(r,fog)}{T(r,f)\{\log T(r,f)\}^k} \geq \frac{\left(\frac{r^2}{4}\right) + O(1)}{\left(\frac{r}{\pi}\right)\{\log r + O(1)\}}$$

$$\text{i.e.}\quad \lim_{r\to\infty} \frac{\log T(r,fog)}{T(r,f)\{\log T(r,f)\}^k} = \infty$$

Theorem 3.8. Let f and g be two entire functions such that $0 < \lambda_f < \infty$ and $0 < \rho_g < \infty$. Also let $0 < \sigma_g < \infty$. Then

$$\limsup_{r\to\infty} \frac{\log^{[2]} M(r,fog)}{\log M(r,g^{(k)})} \geq \frac{\lambda_f}{2^{(k+1)\rho_g}}$$

where K = 0, 1, 2, 3, ...

Proof. Let $0 < \varepsilon < \min\{\lambda_f, \delta_g\}$. Then for a sequence of values of r tending to infinity we obtain.

$$Log M\left(\tfrac{r}{2},g\right) > (\sigma_g - \varepsilon)\left(\tfrac{r}{2}\right)^{\rho_g} \qquad \ldots (6)$$

Again from Lemma 2.1 we get for all sufficiently large values of r,

$$\log^{[2]} M(r, fog) > (\lambda_f - \varepsilon) \log \tfrac{1}{16} + (\lambda_f - \varepsilon) \log M(\tfrac{r}{2}, g) \quad \dots (7)$$

Now for a sequence of values of r tending to infinity it follows from (6) and (7)

$$\log^{[2]} M(r, fog) > (\lambda_f - \varepsilon) \log \tfrac{1}{16} + (\lambda_f - \varepsilon)(\sigma_g - \varepsilon)(\tfrac{r}{2})^{\rho_g} \quad \dots (8)$$

Again by Lemma 2.4 we get for all large values of r,

$$\log M(r, g^{(k)}) < (\sigma_g^{(k)} + \varepsilon) r^{\rho_g} \le (2^{k\rho_g} \sigma_g + \varepsilon) r^{\rho_g} \quad \dots (9)$$

So from (8) and (9) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} M(r, fog)}{\log M(r, g^{(k)})} > \frac{(\lambda_f - \varepsilon) \log \frac{1}{16} + (\lambda_f - \varepsilon)(\sigma_g - \varepsilon)(\frac{r}{2})^{\rho_g}}{(2^{k\rho_g} \sigma_g + \varepsilon) r^{\rho_g}} \quad \dots (10)$$

Since $\varepsilon(>0)$ is arbitrary we get from (10)

$$\limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log M(r, g^{(k)})} \ge \frac{\lambda_f}{2^{(k+1)\rho g}} \quad \dots (13)$$

This proves the theorem.

Theorem 3.9. Let f be meromorphic and g be entire such that $0 < \lambda_{fog} \le \rho_{fog} < \infty$ and $0 < \lambda_g \le \rho_g < \infty$. Then for any positive number A,

$$\frac{\lambda_{fog}}{A\rho_g} \le \liminf_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, g^{(k)})} \le \frac{\lambda_{fog}}{A\lambda_g}$$

$$\le \limsup_{r\to\infty} \frac{\log T(r, fog)}{\log T(r, g^{(k)})} \le \frac{\rho_{fog}}{A\lambda_g} \quad \text{where } k = 0, 1, 2, \ldots\ldots .$$

Proof. From the definition of order and lower order we have for arbitary positive ε and for all large values of r,

$$\log T(r, fog) > (\lambda_{fog} - \varepsilon) \log r \quad \dots (11)$$

and $$\log T(r^A, g^{(k)}) < A(\rho_g^{(k)} + \varepsilon) \log r = A(\rho_g + \varepsilon) \log r \quad \dots (12)$$

Now from (11) and (12) it follows for all large values of r,

$$\frac{\log T(r, fog)}{\log T(r^A, g^{(k)})} > \frac{\lambda_{fog} - \varepsilon}{A(\rho_g + \varepsilon)}$$

As ε (> 0) is arbitary, we obtain

$$\liminf_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, g^{(k)})} \geq \frac{\lambda_{fog}}{A\rho_g} \quad \dots (13)$$

Again for a sequence of values of r tending to infinity,

$$\log T(r, fog) < (\lambda_{fog} + \varepsilon)\log r \quad \dots (14)$$

and for all large values of r,

$$\log T(r^A, g^{(k)}) > A(\lambda_g - \varepsilon)\log r \quad \dots (15)$$

So combining (14) and (15) we get for a sequence of values of r tending to infinity,

$$\frac{\log T(r, fog)}{\log T(r^A, g^{(k)})} < \frac{\lambda_{fog} + \varepsilon}{A(\lambda_g - \varepsilon)}$$

Since ε (> 0) is arbitrary it follows that,

$$\liminf_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, g^{(k)})} \leq \frac{\lambda_{fog}}{A\lambda_g} \quad \dots (16)$$

Also for a sequence of values of r tending to infinity,

$$\log T(r^A, g^{(k)}) < A(\lambda_g + \varepsilon)\log r \quad \dots (17)$$

Now from (11) and (17) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log T(r, fog)}{\log T(r^A, g^{(k)})} > \frac{\lambda_{fog} - \varepsilon}{A(\lambda_g + \varepsilon)}$$

Choosing $\varepsilon \to 0$ We get

$$\limsup_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, g^{(k)})} \geq \frac{\lambda_{fog}}{A\lambda_g} \quad \dots (18)$$

Also for all large values of r,

$$\log T(r, fog) < (\rho_{fog} + \varepsilon)\log r \quad \dots (19)$$

So from (15) and (19) it follows for all large values of r,

$$\frac{\log T(r, fog)}{\log T(r^A, g^{(k)})} < \frac{\rho_{fog} + \varepsilon}{A(\lambda_g - \varepsilon)}$$

As ε (> 0) is arbitrary we obtain

$$\limsup_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, g^{(k)})} \le \frac{\rho_{fog}}{A\lambda_g} \quad \ldots (20)$$

Thus the theorem follows from (13), (16), (18) and (20).

Theorem 3.10. Let f be meromorphic and g be entire such that $0 < \lambda_{fog} \le \rho_{fog} < \infty$ and $0 < \rho_g < \infty$. Then for any positive number A,

$$\liminf_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, g^{(k)})} \le \frac{\rho_{fog}}{A\rho_g} \le \limsup_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, g^{(k)})}$$

where k = 0, 1, 2, ...

Proof. From the definition of order we get for a sequence of values of r tending to infinity,

$$\log T(r^A, g^{(k)}) > A(\rho_g - \varepsilon)\log r \quad \ldots (21)$$

Now from (19) and (21) it follows for a sequence of values of r tending to infinity,

$$\frac{\log T(r, fog)}{\log T(r^A, g^{(k)})} < \frac{\rho_{fog} + \varepsilon}{A(\rho_g - \varepsilon)}$$

As ε (> 0) is arbitrary we obtain,

$$\liminf_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, g^{(k)})} \le \frac{\rho_{fog}}{A\rho_g} \quad \ldots (22)$$

Again, for a sequence of values of r tending to infinity,

$$\log T(r, fog) > (\rho_{fog} - \varepsilon)\log r. \quad \ldots (23)$$

So combining (12) and (23) we get for a sequence of values of r tending to infinity,

$$\frac{\log T(r, fog)}{\log T(r^A, g^{(k)})} > \frac{\rho_{fog} - \varepsilon}{A(\rho_g + \varepsilon)}.$$

Since ε (> 0) is arbitrary it follows that,

$$\limsup_{r\to\infty}\frac{\log T(r,fog)}{\log T(r^{A},g^{(k)})}\geq\frac{\rho_{fog}}{A\rho_{g}} \qquad \ldots (24)$$

Thus the theorem follows from (22) and (24).

In the next theorem we estimate the lower order of a composite meromorphic function whose left factor is of zero order.

Theorem 3.11. Let f be a meromorphic function of order zero and g be an entire function such that $\lambda_g<\infty$. If $\lambda_f^*<\infty$ then $\lambda_{fog}<\infty$.

Proof. By Lemma 2.2 and the inequality $T(r,g)\leq\log^{+}M(r,g)$ we obtain for all sufficiently large values of r,

$$\frac{\log T(r,fog)}{\log^{[2]}M(r,g)}\leq\frac{\log T(M(r,g),f)+O(1)}{\log^{[2]}M(r,g)}$$

$$\text{or,}\quad \liminf_{r\to\infty}\frac{\log T(r,fog)}{\log^{[2]}M(r,g)}\leq\lambda_f^*<\infty \qquad \ldots (25)$$

by the given condition.

Now by Lemma 2.5 and (25) it follows that $\lambda_{fog}<\infty$. This proves the theorem.

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