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ON THE ORDER AND TYPE OF DIFFERENTIAL MONOMIALS

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ABSTRACT : In the paper we study the relation between the order (type) of a transcendental meromorphic function and that of a differential monomial generated by it.

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1. INTRODUCTION AND DEFINITIONS

Let f be a transcendental meromorphic function defined in the open complex plane C and n_0, n_1, ...n_k, be non negative integers such that $\sum_{i=0}^{k} n_i \geq 1$. We call $p[f] = af^{n_0}(f^{(1)})^{n_1} ......(f^{(k)}),^{n_k}$ where T(r,a) = S(r,f) to be a differential monomial generated by f. The numbers $\gamma_p = \sum_{i=0}^{k} n_i$ and $\Gamma_p = \sum_{i=0}^{k} (i+1)n_i$ are respectively called the degree and weight of P[f].

In the paper we establish the relation between the order (type) of P[f] and f.

The following definitions are well known.

Definition 1.1. [4] For a $\in$ C $\cup$ {∞} we denote by n(r,a; f$|$=1) the number of simple zeros of f-a in $|z| \leq$ r. N(r,a; f$|$=1) is defined in terms of n(r,a; f$|$=1) in the usual way. Also we put

$$\delta_1(a;f) = 1 - \lim_{r\to\infty} \sup \frac{N(r,\ a;\ \mathrm{f}\ |=1)}{\mathrm{T(r,\ f)}}$$

Yang [3] proved that there exists at most a denumerable number of complex numbers a $\in$ C $\cup$ {∞} for which δ_1(a; f) > 0 and $\sum_{a\in C\cup\{\infty\}} \delta_1(a;\ \mathrm{f}) \leq 4$

Definition 1.2. The order ρ_f and lower order λ_f of a meromorphic function f is defined as

$$\rho_f = \lim_{r\to\infty} \sup \frac{\log T(r, f)}{\log r}$$

$$\text{and} \quad \lambda_f = \lim_{r\to\infty} \inf \frac{\log T(r, f)}{\log r}$$

if f is entire then

$$\rho_f = \lim_{r\to\infty} \sup \frac{\log^{[2]} M(r, f)}{\log r}$$

$$\text{and} \quad \lambda_f = \lim_{r\to\infty} \inf \frac{\log^{[2]} M(r, f)}{\log r}$$

Definition 1.3. The hyper order $\overline{\rho}_f$ and hyper lower order $\overline{\lambda}_f$ of a meromorphic funciton f is defined as

$$\overline{\rho}_f = \lim_{r\to\infty} \sup \frac{\log^{[2]} T(r,f)}{\log r} \quad \text{and} \quad \overline{\lambda}_f = \lim_{r\to\infty} \inf \frac{\log^{[2]} T(r,f)}{\log r}$$

If f is entire then one can easily verify that

$$\overline{\rho}_f = \lim_{r\to\infty} \sup \frac{\log^{[3]} M(r,f)}{\log r}$$

$$\text{and} \quad \overline{\lambda}_f = \lim_{r\to\infty} \inf \frac{\log^{[3]} M(r,f)}{\log r}$$

Definition 1.4. The type of σ_f a meromorphic function f is defined as

$$\sigma_f = \lim_{r\to\infty} \sup \frac{T(r,f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

In the paper we do not explain the standard notations of value distribution theory as those are available in [1].

2. LEMMA

In this section we present a lemma which will be needed in the sequel.

Lemma 2.1. [2] Let f be of finite order or of non-zero lower order. If $\sum_{a \in C \cup \{\infty\}} \delta_1(a, f) = 4$

then $\lim_{r \to \infty} \frac{T(r, P[f])}{T(r, f)} = \Gamma_p - (\Gamma_p - \gamma_p)\, \theta(\infty, f)$.

3. THEOREMS

In this section we present the main results of the paper.

Theorem 3.1. Let f be of positive finite order and $\sum_{a \in C \cup \{\infty\}} \delta_1(\alpha, f) = 4$ Then the order of P[f] is same as that of f and type of P[f] is $\{\Gamma_p - (\Gamma_p - \gamma_p)\, \theta(\infty, f)\}$ times that of f.

Proof. Let ρ_1, ρ_2 be the orders and τ_1, τ_2 be the types of f and P[f] respectively. Then by Lemma 2.1 we get

$$\rho_1 = \limsup_{r \to \infty} \frac{\log T(r, P[f])}{\log r}$$

$$= \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} \cdot \frac{\log T(r, P[f])}{\log T(r, f)}$$

$$\leq \rho_1 \limsup_{r \to \infty} \frac{\log T(r, P[f])}{\log T(r, f)}$$

$$= \rho_1 \limsup_{r \to \infty} \frac{\log \frac{T(r, P[f])}{T(r, f)} + \log T(r, f)}{\log T(r, f)}$$

$$= \rho_1 \qquad \cdots (1)$$

Again $\rho_1 = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}$

$$\leq \rho_2 \limsup_{r \to \infty} \frac{\log \frac{T(r, f)}{T(r, P[f])} + \log T(r, P[f])}{\log T(r, [f])}$$

$$= \rho_2 \qquad \cdots (2)$$

From (1) and (2) we get $\rho_1 = \rho_2$.

Now by Lemma 2.1 we see that

$$\tau_2 = \limsup_{r\to\infty} \frac{T(r,P[f])}{r^{\rho_2}}$$

$$= \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_1}} \cdot \lim_{r\to\infty} \frac{T(r,P[f])}{T(r,f)}$$

$$= \left\{\Gamma_p - (\Gamma_p - \gamma_p)\,\theta(\infty, f)\right\}\tau_1.$$

This proves the theorem.

Theorem 3.2. Let f be of finite order or of non zero lower order. If $\sum_{a\in C\cup\{\infty\}} \delta_1(a;\, f) = 4$ then the lower order of P [f] is same as that of f.

The proof is omitted.

Theorem 3.3. Let f be of finite order or of non zero lower order. Also let $\sum_{a\in C\cup\{\infty\}} \delta_1(a;\, f) = 4$ Then the hyper order of P [f] is same as that of f.

Proof. Let $\bar{\rho}_1$, $\bar{\rho}_2$ be the hyper orders of f and P [f] respectively.

Now in view of Lemma 2.1, $\lim_{r\to\infty} \frac{\log^{[2]} T(r,\, P[f])}{\log^{[2]} T(r,f)}$ exists and is equal to 1.

Thus we get,

$$\bar{\rho}_2 = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,\, P[f])}{\log r}$$

$$= \limsup_{r\to\infty} \left\{ \frac{\log^2 T(r,f)}{\log r} \cdot \frac{\log^{[2]} T(r,\, P[f])}{\log^2 T(r,f)} \right\}$$

$$= \limsup_{r\to\infty} \frac{\log^2 T(r,f)}{\log r} \cdot \lim_{r\to\infty} \frac{\log^{[2]} T(r,\, P[f])}{\log^{[2]} T(r,f)}$$

$$= \bar{\rho}_1.$$

Theorem 3.4. Let f be of finite order or of non zero lower order and $\sum_{a \in C \cup (\infty)} \delta_1(a; f) = 4$. Then the hyper lower orders of P[f] and f are same.

The proof is omitted.

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