Jour. Pure Math., Vol. 19, 2002, pp. 41-47

s–A RADICAL OF A NEAR-RING WITH CHAIN CONDITION

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ABSTRACT : This papaer deals with results on near-rings with ascending chain conditions on annihilators having no infinite direct sum of ideals (subgroups) and with parts satisfying the acc or the dcc on its substructures. The s-A radical of such parts gives the corresponding factor near-ring a similar structure. A countable s-A radical has no essential extension in N. A minimal ideal satisfying the acc on its left N-subgroups satisfies the descending chain conditions. In some cases the s-A radical coincides with the right annihilator of the left annihilator of the s-A radical coincides with the right annihilator of the left annihilator of the s-A radical. If the socle of N is with dcc on its N-subgroups than N also inherits the same character.

Mathematics Subject Classification (AMS 2000) : 16 Y 30, 16 P 40, 16 P 60

Key words and phrases : Near-rings, s-A radical, Essential N-subgroup, Socle, Annihilators, chain conditions.

1. INTRODUCTION

The study of near-rings with chain conditions is well developed [2,3,6,8,9] but the modifications on conditions such as ascending chain or descending chain on substructures of near-rings have left scopes for further studies. In our continuous afford we have studied near rings satisfying acc on annihilator[2,3,6]. The objective of this work is on near rings with acc on annihilators having no infinite direct sum of ideals(subgroups) and with parts satisfying the acc or the dcc on its substructures. The utility of such substructures motivates one to study the strictly Artinian radical (s A radical) of such a near-ring. Slicing a radical out from near-ring will yield a simpler and amenable near-ring. Further information about the original near-ring from the structure of the sliced radical can be also expected.

In [3] we introduced the concept of such a particular type of radical substructure, termed as striclty Artinian radical. We define a strictly Artinian radical, the s A radical as the sum s-A(N) of all left ideals of N, each satisfying the dcc on its left N-subgroups. This paper deals with results on this radical character of a strongly semiprime near-ring N satisfying the conditions :

(i) N has no infinite independent family of left N- subgroups of it

and (ii) N satisfies the acc on left annihilators of subsets of N.

Here we prove the inheritance of dcc of a near-ring modulo the left annhilator of a countable s A radical. The s A radical has no countable extensions in N. If N is with acc on its left N-subgroups and s-A(N) is an invariant subnear-ring of N then s-A(N) = r(1(s-A(N))). If the socle of N is with dcc on its N-subgroups then N is also with dcc on its left N-subgroups.

2. DEFINITIONS AND NOTATIONS

All basic definitions used in the paper are referred to Pilz [9]. Throughout the paper N stands for a zero symmetric right dgnr with unity. E will denote a left N-group $_N E$. A left N-subgroup A of N will mean an N-subgroup of $_N N$ and left ideal of N will mean an idal of N will mean an ideal of $_N N$. Let A and B be two left N-subgroups of N, with A $\subseteq$ B then A is said to be an *essential left N-subgroup* of B, if any N-subgroup C ($\neq$ 0) of B has non zero intersection with A. If A is an essential left N-subgroup of B we say B is an *essential extension* of A in N. A left N-subgroup A of N is a *weakly essential left N-subgroup* of N if for any left ideal I ($\neq$ 0) of N, A $\cap$ I $\neq$ (0). It is clear that an essential left N-subgroup of N is a weakly essential left N-subgroup of it. But the converse is not true.

In the near-ring N = {0, a, b, c, x, y} defined as in (H) S^3, [9, p.342 (37)] we note that {0,a}, {0,b}, {0,c}, {0,x,y} are proper non zero left N-subgroups where {0,x,y} is a left ideal. This is not an essential left N-subgroup though it is weakly essential. A left ideal which is weakly essential as a left N-subgroup is an essential left ideal of N. Near-ring N is left singular if no non-zero element of N annihilates any essential left N-subgroup of N from the right. An element x $\in$ N is regular if there is an element y $\in$ N such that xyx = x and N is regular if each element of N is regular. Near-ring N is strongly semi prime if it has no nonzero nilpotent subset. It is easy to see that a regular near-ring is always strongly semiprime. And a partial converse of it is seen in Oswald [Corollary 4., [7]]. The collection of all maximal left annihilators of the type (P =) 1 (A), where A is a nonzero left N-subgroup of N is denoted by Γ. We confine our discussion to the subfamily P ($\subseteq \Gamma$) ={P | P = 1(A), where A is a nonzero invariant subnear-ring of N}. In the near-ring N = {0,a,b,c} under addition and multiplication defined as in [9, p.340(7)] the nonzero left N-subsets are {0,a}, {0,b}, {0,a,b} and N. Clearly their left annihilators are {0}, {0,a} and {0,b}. Thus Γ = {{0,a}, {0,b}}. On the other hand, nonzero invariant sub near-rings are {0,a}, {0,b}} and N and their left annihilators are {0}, {0,a}, {0,b}. Here P = {{0,a}, {0,b}}. Thus this example shows P $\subseteq \Gamma$. In near-ring N = {0,a,b,c} [[1], 2.1, 10], B = {0,b} is the only proper left ideals of N and clearly BN = NB = B. If N = {0,a,b,c} is a near-ring as defined in [[1],2.2,13] it is seen that A = {0,a}, B = {0,b} and C = {0,c} are proper left ideals of N. Clearly, AB = B, BA = (0), BC = (0), CB = B, AC = C, CA = A, A^2 = A, B^2 = (0) and C^2 = C.

The above examples show that any finite product of (left) ideals is an (left) ideal. In case of strongly regular near-ring we get that finite product of (left) ideals is an (left) ideal as in [5]. In this sense such near-rings are termed as (left) *ideal closed near-rings*, in short (1,i) closed *near-rings*. Thus a strongly regular near-ring is (1,i) closed. But the near-rings given above are not strongly regular though the product of two ideals is again an ideal. So, an (1,i) closed near-ring need not be strongly regular. N is called (1,i) *closed* if any finite product of left ideals is also a left ideal [3]. We define *s A redical of* N as the sum s-A(N) of all left ideals of N, each satisfying the dcc on its left N-subgroups.

If N is with dcc on its left N-subgroups then s-A(N) = N and if N has non zero left ideals with dcc on its left N-subgroups then s-A(N) = (0).

$N = \begin{bmatrix} Z & Q \\ Q & Q \end{bmatrix}$ is a near-ring with acc on its left annihilators and it has no infinite direct sum of left ideals. If e_{ij} denotes the matrix with 1 in the (i,j)th position and 0's elsewhere, then the left ideals Qe_{21}, Qe_{22}, of N are with dcc on its N-subgroups (where Qe_{21}, Qe_{22}, denote the set of all 2X2 matrices with an arbitrary element of Q in the (1,2)th-position and (2,2)th-position respectively). Here s-A(N) = $Qe_{21} + Qe_{22}$.

Socle of E denoted by Soc(E) is the sum of all simple ideals of E. Soc(E) is also the intersection of all essential ideals of E [66]. N-group E is *finitely* N-*cogenerated* in case for every set of N-subgroups of E, $\cap$ A = 0 implies $\cap$ F = 0 for some finite F $\subseteq$ A.

Now we note the following.

Note 2.1. If N is strongly semiprime near-ring with acc on left annihilators then N is left non singular.

Note 2.2. If N satisfies the acc on annihilators than $Z_l(N)$ is a nil invariant subset of N.

Note 2.3. If for some nonzero left N-subgroup A of a strongly semiprime near-ring N, P = l(A) is a maximal left annihilator then P is a minimal strongly prime ideal.

Note 2.4. If for some nonzero left N-subgroup A of a strongly semiprime near-ring N, P = l(A) is a maximal left annihilator then P is a minimal strongly prime ideal.

3. PRELIMINARIES

Lemma 3.1. If N satisfies the acc on left annihilators, then $Z_l(N)$ is nilpotent.

Proof. We write $Z = Z_l(N)$. As $Z \supseteq Z^2 \supseteq \ldots$, we get $l(Z) \subseteq l(Z^2)$ ...Since N satisfies the acc on left annihilators, we have a $t \in Z^+$ such that $l(Z^t) = l(Z^{t+1}) = \ldots$

If $Z^{t+1} \neq (0)$ then there is an element $a \in Z$ such that $aZ^t \neq (0)$.

We choose $l(a)$ as large as possible with $aZ^t \neq (0)$

Now $b \in Z$ gives $1(b) \subseteq_e N$ which implies $1(b) \cap Na \neq (0)$ and so $na \in 1(b)$, $na \neq 0$ for some $n \in N$. Z being invariant N-subset of N, we have $ab \in Z$. Also $1(a) \subseteq 1(ab)$.

Now $na \neq 0$ and $nab = 0$ gives $1(a) \subset 1(ab)$. So, the choice of a gives that $abZ^{t} = (0)$.

Thus, $aZ^{t+1} = (0)$, a contradiction and hence, $Z^{t+1} = (0)$ implies Z is nilpotent.

Lemma 3.2. If $A \subseteq_e E$, $B \subseteq_e E$ then $A \cap B \subseteq_e E$

Proof. Let D ($\neq 0$) be an N-subgroup of E. As $A \subseteq_e E$, $A \cap D \neq (0)$. Also $A \cap D$ is an N-subgroup of E and $B \subseteq_e E$ gives $(A \cap D) \cap B \neq (0)$ implies $(A \cap B) \cap D \neq (0)$. Thus $A \cap B \subseteq_e E$.

Lemma 3.3. If A, B are left N-subgroups of N with $A \subseteq B$ and A, B/A satisfy the dcc on its N-subgroups then B is also with dcc on its N-subgroups.

Proof. Let $B_1 \supseteq B_2 \supseteq \ldots \ldots \ldots$ be a descending chain of N-subgroups of B. Since B/A is with dcc on its N-subgroups, there is an $r \in Z^+$ such that $B_r + A = B_{r+i} + A$. Also A is with dcc on its N-subgruops implies there is an integer $s \in Z^+$ such that $A \cap B_s = A \cap B_{s+1}$, for $A \cap B_1 \supseteq A \cap B_2 \supseteq \ldots$ is a descending chain of N-subgroups of A. Now, if $t = \max(r,s)$ then $B_t = B_t \cap (B_t + A)$ implies $B_t \cap (B_{t+1} + A) = B_{t+1} + (B_t \cap A) = B_{t+1} + (B_{t+1} \cap A) = B_{t+1}$. Thus, B is with dcc on its N-subgroups.

Lemma 3.4. If I is a finitely generated N-group and N is with dcc on its left N-subgroups then I is with dcc on its N-subgroups.

Proof. As I is finitely generated, we have finite set $\{i_1, i_2, \ldots I_t\} \subseteq I$, such that $Ni_1 \oplus \ldots \oplus Ni_t \to I \to 0$ is exact. Since N is with dcc on its N-subgroups, $N^F = \oplus N$ is also so. As I is isomorphic to a factor of N^F we get I is with dcc on its N-subgroups.

Lemma 3.5. If N-group E is with dcc on its N-subgroups then E is finitely N-cogenerated.

Proof. Let A be the collection of N-subgroups of E with dcc on its N-subgroups. If $\cap A = 0$ and $B = \{ \cap F \mid F$ is a finite sub collection of $A\}$ then B has a minimal element. Since $\cap A = 0$, it follows that this minimal element must be zero. So for some finite subclass F of A, $\cap F = 0$. Thus the result follows.

Lemma 3.6. If N-group E is with dcc on its N-subgroups then Soc E is essential in E.

Proof. Let B be an ideal of E with $(\mathrm{Soc}E) \cap B = 0$. Now Soc E = intersection of all essential ideals of E. Since E is finitely cogenerated by 3.5. So, there exists essential ideals $I_1, I_2, \ldots I_k$ of E with $I_1 \cap I_2 \cap \ldots \cap I_k \cap B = 0$. Also $I_1, I_2, \ldots I_k$ are essential in E. By 3.1.9., $I_1 \cap I_2 \cap \ldots \cap I_k \subseteq_e E$. Thus $B = 0$ giving thereby Soc $E \subseteq_e E$.

As in ring theory we get the following Lemma.

Lemma 3.7. If N-group E is with dcc on its N-subgroups and L is an N-subgroup of E then SocL = L $\cap$ SoeE.

Lemma 3.8. If I is a left ideal of a strongly semiprime near-ring N and if N/1(I) (= $\overline{N}$) satisfies the dcc on its left $\overline{N}$-subgroups then I also satisfies the dcc on its N-subgroups.

Proof. If $A_1 \supseteq A_2 \supseteq \ldots$ is a descending chain of N-subgroups of I then we get a descending chain $Q_1 \supseteq Q_2 \supseteq \ldots$ of left $\overline{N}$-subgroups of $\overline{N}$ where $Q_i = \{a_i + 1(I) \mid a_i \in A_i\}$. As $\overline{N}$ is with dcc on its left $\overline{N}$-subgroups so, $Q_t = Q_{t+1}$, for some $t \in Z^+$. Now $A_i \cap 1(I) \subseteq A_i$ and 1(I) implies $A_i \cap 1(I) \subseteq I$. So $[(A_i \cap 1(I)]^2 \subseteq 1(I)I = (0)$ which gives $[A_i \cap 1(I)]^2 = (0)$. And N being strongly semiprime $A_i \cap 1(I) = (0)$. Thus for all i, $A_i \cap 1(I) = (0)$. So, $Q_t = Q_{t+1} \Rightarrow A_t = A_{t+1} = \ldots$ The result follows.

For two left ideals A, B of N, (A + B) / B and A / (A $\cap$ B) are isomorphic. If A, B are with dcc on their N-subgroups A $\cap$ B is also so. Thus (A + B) / B is also with dcc on its N-subgroups. Hence A + B is with dcc on its N-subgroups. Thus we easily get.

Lemma 3.9. [3] s-A(N) is a left ideal of N and it satisfies the dcc on its N-subgroups.

In case of $N = \begin{bmatrix} Z & F \\ 0 & Z \end{bmatrix}$, $F = Z/2Z$, it satisfies the condition that for ideals I and X, if N / X is with dcc on N-subgroups then there is an ideal Y such that N / Y is also is so.

The following result applies to near-ring N which satisfies the property that if N / X is with dcc on its N-subgroup then so is N / Y. The property that N is a (1,i) closed near-ring plays an important role.

Lemma 3.10. [3] If N is a strongly semiprime near-ring as above and is with distributively generated left annihilators, s-A(N) is minimal and countable invariant sub near-ring then for any minimal strongly prime ideals of N with dcc on its left N-subgrups, s-A(N) + 1(s-A(N)) contains a non zero divisor.

4. MAIN RESULTS

We now prove our main results on s-A radical of a strongly semiprime near-ring.

Theorem 4.1. Let N be a strongly semiprime near-ring. If s-A(N) is countable then s-A(N) has no countable essential extension in N.

Proof. Let B be a proper countable left N-subgroup of N such that s-A(N) is an essential left N-subgroup of B. So B has an essential left N-subgoup which satisfies the dcc on its left N-subgroups. Since B is countable, by Lemma 3.1[6] there exists a finite $S = \{y_1, \ldots Y_t\}$ ($\subseteq$B) such that 1(B) = 1(S).

We define $\varphi : N/1(B) \rightarrow Ny_1 \oplus \ldots\ldots\ldots \oplus Ny_t$ such that $\varphi(\bar{x}) = (xy_1, \ldots xy_t)$. For any $\bar{a}, \bar{b} \in N/1(B)$ we get, $\varphi(\bar{a} + \bar{b}) = \varphi(\bar{a}) + \phi(\bar{b})$ and $\varphi(n\,\bar{a}) = n\varphi(\bar{a})$, $n \in N$. Also $\varphi(\bar{a}) = \varphi(\bar{b})$ gives $(ay_1, \ldots, ay_t) = (by_1, \ldots, by_t)$ and so $(a-b)y_i = 0$. Hence, $(a-b) \in 1(y_i)$, for $i = 1, \ldots, t$ which implies $a-b \in 1(y_1) \cap \ldots \cap 1(y_t) = 1(B)$. Thus, $\bar{a} = \bar{b}$. And so φ being an N-monomorphism, $N/1(B))$ can be embedded in $Ny_1 \oplus \ldots\ldots\ldots \oplus Ny_t$ as an N-group and thus $N/1(B)$ is embedded in the direct sum of a finite number of copies of B. Also $N/1(B)$ has an essential left N-subgroup (s-A(N) $\oplus$ 1(B)) / 1(B) which satisfies the dcc on its left N-subgroups, as s-A(N) is with dcc. Thus $N/1(B)$ has an essential left subgroup with dcc on its left N-subgroups. By Lemma 3.6 and 3.7, $M = \mathrm{Soc}\,(N/1(B))$ is an essential left N-subgroups of $N/1(B)$. If $x \in 1(M)$, for some $x \in N/1(B)$ then $xM = (0)$. Since N is strongly semiprime, $1(M) \subseteq r(M)$. Thus $xM = Mx = 0$ implies $x \in Z_l(N/1(B))$ and so $1(M) \subseteq Z_l(N/1(B))$.

If $\mathcal{N}$ is the nil radical of $N/1(B)$ then it is the largest nil ideal which gives $Z_l(N/1(B)) \subseteq \mathcal{N}$ and thus $1(M) \subseteq \mathcal{N}$. By what we have got, M is with dcc on its left $N/1(B)$ ($= \overline{N}$)-subgroups. So $N/1(M)$ is with dcc on its $\overline{N}$-subgroups and thus $\overline{N}/\mathcal{N}$ is with dcc on its $\overline{N}$-subgroups. So by Lemma 3.8, B satisfies the dcc on its left N-subgroups. Therefore $B \subseteq$ s-A(N) which gives s-A(N) = B, a contradiction. Thus, s-A(N) has no countable essential extension in N.

Theorem 4.2. Let N be a strongly semiprime near-ring. If s-A(N) is countable then the N-group $N/1$(s-A(N)) satisfies the dcc on its N-subgroups.

Proof. s-A(N) being countable, by Lemma 3.1[6], there is a finite set $S = \{y_1, \ldots y_t\}$ s-A(N) such that $1(\text{s-A(N)}) = 1(y_1) \cap \ldots \cap 1(y_t) = 1(S)$. As in the above theorem $N/1$(s-A(N)) can be embedded in $Ny_1 \oplus \ldots\ldots\ldots \oplus Ny_t$ as an N-group. Thus $N/1$(s-A(N)) satisfies the dcc on its N-subgroups.

Theorem 4.3. If N is as above then s-A(N /s-A(N)) = $(\bar{0})$.

Proof. Let I be a left ideal of N each such that s-A(N) I and I / s-A(N) be with dcc on its left (N / s-A(N) =) $\overline{N}$ subgroups. If J /s-A(N) is an $\overline{N}$-subgroup of I / s-A(N) then it is an N-subgroup of I / s-A(N). Hence I / s-A(N) is with dcc on its left N-subgroups. By Lemma 3.9, s-A(N) is with dcc on its left N-subgroups. Thus by Lemma 3.3, I satisfies the dcc on its N-subgroups. So, $I \subseteq$ s-A(N) which gives I = s-A(N). Therefore N /s-A(N) does not have any nonzero left ideal with dcc on its N-subgroups. This implies s-A(N /s-A(N)) $= (\bar{0})$.

Theorem 4.4. Let N be a strongly semiprime near-ring. If s-A(N) is an invariant sub near-ring of N then s-A(N) = r(1(s-A(N)).

Proof. We denote s-A(N) by K. By theorem 4.2 $N/1(K)$ satisfies the dcc on its N-subgroups. We consider the map $N/1(K) \times r(1(K)) \rightarrow r(1(K))$ by $\bar{x}, \alpha \rightarrow x\alpha$. The map is well defined. For if $\bar{x} = \bar{y}$ then $x-y \in 1(K)$. Now $\alpha \in r(1(K))$ gives $1(K)\alpha = (0)$ and so $x\alpha = y\alpha$. Also $(\bar{x}+\bar{y})\alpha = \bar{x}\alpha + \bar{y}\alpha$, $(\bar{x}\;\bar{y})\alpha = \bar{x}\,(\bar{y}\alpha)$ and $\bar{1}\alpha = \alpha$ So, r(1(K)) is a left $N/1(K)$ group. As N satisfies the acc on its N-subgroups, r(1(K)) is with dcc on its $N/1(K)$-subgroups. So, r(1(K)) satisfies the dcc on its N-subgroups. Thus $r(1(K)) \subseteq K$. Also if $x \in K$ then $1(K)\,x = (0)$ implies $x \in r(1(K))$. So, $K \subseteq r(1(K))$. Thus $K = r(1(K))$.

Theorem 4.5. Let N be a strongly semiprime near-ring as in Lemma 3.10. If Soc(N) is essential in N and Soc(N) is with dcc on its N-subgroups then N is also with dcc on its N-subgroups.

Proof. We set K = s-A(N). By lemma 3.10, K + 1(K) contains a non-zero divisor c. Using Lemma 3.2 [6] we get Nc is an essential left N-subgroup of N. If $x \in r(Nc)$ then $Ncx = 0$. Since N is strongly semiprime with acc on annihilators, by Note 2.1, $Z_1(N) = 0$. So, Nc being essential subgoup of N we get $x = 0$. Thus, $r(Nc) = (0)$. Also, $Nc \subseteq K + 1(K)$ gives $r(K + 1(K)) \subseteq r(Nc) = (0)$ and so $r(K + 1(K)) = (0)$. Since, $r(K) \cap r(1(K)) \subseteq r(K + 1(K)) = (0)$. So we get $r(K) \cap r(1(K)) = (0)$. Hence by Theorem 4.4, $r(K) \cap K = (0)$. Since Soc (N) is with dcc on its N-subgroups and Soc(N) is an essential left ideal of N we get $Soc(N) \subseteq K$ and thus $Soc(N) \cap r(K) = 0$. Hence $r(K) = (0)$. Now, $K = r(1(K)) = r(r(K)) = r(0) = N$. Thus, N is with dcc on its left N-subgroups.

ACKNOWLEDGEMENT

The author is thankful to Prof. K. C. Chowdhury, Department of Mathematics of Gauhati University for his suggestion and help in preparing this manuscript.

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