Jour. Pure Math., Vol. 19, 2002, pp. 73-79

PRODUCT PSEUDO ALGEBRAIC SPACES

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ABSTRACT : If (X,T_1) and (Y,T_2) be two p-topological spaces, then $T^* = \{A_1 \times A_2 : A_1 \in T_1, A_2 \in T_2\}$ is defined to be a Product p-topology on $X \times Y$. Finally Product p-topological space and Product p-a space are established and some of its properties are discussed here.

Key words and phrases : Pseudo topological space, Product pseudo topological space and Product pseudo algebraic space are denoted by p-topological space, Product p-topological space and Product p-a space.

1. INTRODUCTION

The aim of this paper is to introduce the notion of Pseudo algebraic spaces. A pseudo algebraic space is defined to be a non-empty set having two types of structures—a Pseudo topological structure and a Pseudo algebraic structure. We have introduced the notion of sub p-a spaces, keeping it in mind that notions of topological subspaces and subgroups go together. We have introduced a special kind of mapping from one p-a space to another. These mappings are counter-part of homomorphism in algebra and continuous functions in topology. We call these mappings p-a homomorphisms. One of the purposes of the topic is to study the elementary properties of these p-a homomorphisms and form a basis for further study of p-a spaces.

Our special interest would be on Product p-topological space. We have introduced a p-a structure on Product p-topological space. A Product p-topological space equipped with a p-a structure is called a Product p-a space.

2. PRELIMINARIES

Definition 2.1. Let X be a non-empty set and T a class of subsets of X such that

(i) $X \in T$

(ii) there exists an $A_0 \in T$ such that $A_0 \subseteq A$ for every $A \in T$

(iii) any finite intersection of members of T is a member of T.

The class T is called a Pseudo topology (p-topology) and the pair (X,T) is called a p-topological space. When there is no scope for confusion, X may be simply called a p-topological space. The members of T are called Pseudo open set (p-open sets) of X. A set A_0 with the property (ii) is called a minimal p-open set. In a p-topological space, there is one and only one minimal p-open set. Therefore, a minimal p-open set is referred as the minimal p-open set.

Definition 2.2. A p-topological space (X,T) is said to have a Pseudo algebraic structure (p-a structure) if there exists a Pseudo algebraic function,

$$\alpha : P^X \times P^X \to P^X \quad (P^X \text{ is the power set of } X)$$

satisfying the following conditions.

(i) $\alpha(\alpha(A,B), C) = \alpha(A, \alpha(B,C))$, $A,B,C, \in P^X$

(ii) $\alpha(A,B) \in T$ if $\alpha(A,B) = \alpha(B,A)$ for $A, B \in T$

(iii) if $A_1 \subseteq A$, $B_1 \subseteq B$ then $\alpha(A_1, B_1) \subseteq \alpha(A,B)$

(iv) $\alpha(A_0,A) = \alpha(A,A_0)$, $A \in P^X$, where A_0 is the minimal p-open set.

We say that the triple (X,T, α) is a p-topological space with a p-a structure α or simply a Pseudo algebraic space (p-a space).

Definition 2.3. A subset A in a p-a space (X,T, α) is called a normal set if $\alpha(A,Y) = \alpha(Y,A)$, $\forall\, Y \in P^X$.

Definition 2.4. The p-topology T of a p-a space (X,T, α) is said to be normal if every p-open set is normal and a p-a space is said to be normal if its p-topology is normal.

Definition 2.5. The p-topology T′ on Y which is a non-empty subset of X defined by

$T' = \{ A \cap Y : A \in T\}$ is called the relative p-topology on Y and the p-topological space (Y, T′) is called a sub p-topological space of (X,T).

Definition 2.6. A sub p-topological space (Y,T′) is called a sub p-a space of a p-a space (X,T, α) with a p-a structure α' if α induces a p-a function α' on P^Y such that

(i) $\alpha'(A,B) = \alpha(A,B)$, $A,B \in P^Y$

(ii) $\alpha'(A,B) \in T'$ if $\alpha'(A,B) = \alpha'(B,A)$ for $A,B \in T'$

and (iii) $\alpha'(A'_0,A) = \alpha(A,A'_0)$, $A \in P^Y$, where A'_0 is the minimal p-open set in T′.

Definition 2.7. Let (X,T) and (Y,T^*) be two p-topological spaces. Let f be a maping from X to Y. We say that the mapping f is p-continuous if $f^{-1}(A^*) \in T$ whenever $A^* \in T^*$ and is called p-open if $f(A) \in T^*$ whenever $A \in T$.

Definition 2.8. Let (X,T,α) and (Y,T^*,β) be two p-a spaces.

A function $f : X \to Y$ is called a p-a homomorphism if it is such that

(i) f is both p-open and p-continuous

(ii) $f(\alpha(A,B)) = \beta(f(A),f(B))$, $A, B \in P^X$

and (iii) $\alpha(f^{-1}(A^*), f^{-1}(B^*)) = f^{-1}(\beta(A^*,B^*))$, $A^*, B^* \in P^Y$.

3. PRODUCT PSEUDO ALGEBRAIC SPACES

Now we establish the Product p-topological space and Product p-a space.

Proposition 3.1. Let (X,T_1) and (Y,T_2) be two p-topological spaces. Let $T^* = \{A_1 \times A_2 : A_1 \in T_1, A_2 \in T_2\}$. Then T^* is a p-topology on $X \times Y$.

Proof.

(i) $X \in T_1$, $Y \in T_2 \Rightarrow X \times Y \in T^*$

(ii) Let A_0 and B_0 be the minimal p-open sets of T_1 and T_2 respectively. Then $A_0 \times B_0 \in T^*$. is the minimal p-open set of T^*.

(iii) Let $A_1 \times A_2$, $B_1 \times B_2 \in T^*$

$(A_1 \times A_2) \cap (B_1 \times B_2) = (A_1 \cap B_1) \times (A_2 \cap B_2) \in T^*$

since $A_1 \in T_1$, $B_1 \in T_1 \Rightarrow A_1 \cap B_1 \in T_1$

and $A_2 \in T_2$, $B_2 \in T_2 \Rightarrow A_2 \cap B_2 \in T_2$

T^* is a p-topology on $X \times Y$.

Remark 3.2. T^* is called the Product p-topology on $X \times Y$ and $(X \times Y, T^*)$ is called the Product p-topological space.

Proposition 3.3. Let (X,T_1,α_1) and (Y,T_2,α_1) be two p-a spaces where α_1, and α_2 are p-a structure on X and Y respectively.

Let $T^* = \{ A_1 \times A_2 : A_1 \in T_1, A_2 \in T_2\}$ be a p-topology on $X \times Y$.

Let $\alpha^* : P^{X\times Y} \times P^{X\times Y} \to P^{X\times Y}$ be such that

$\alpha^*(A_1 \times A_2, B_1 \times B_2) = \alpha_1(A_1,B_1) \times \alpha_2(A_2,B_2)$

Then $(X \times Y, T^*, \alpha^*)$ is a p-a space.

Proof. We show that α^* is a p-a structure on $X \times Y$

(i) $\alpha^*(\alpha^*(A_1 \times A_2, B_1 \times B_2), C_1 \times C_2)$

$= \alpha^*(\alpha_1(A_1,B_1) \times \alpha_2(A_2,B_2), C_1 \times C_2)$

$= \alpha_1(\alpha_1((A_1,B_1), C_1) \times \alpha_2(\alpha_2((A_2,B_2),C_2))$

$= \alpha_1(A_1,\alpha_1(B_1,C_1)) \times \alpha_2(A_2,\alpha_2(B_2,C_2))$

$= \alpha^*(A_1 \times A_2, \alpha_1(B_1,C_1) \times \alpha_2(B_2,C_2))$

$= \alpha^*(A_1 \times A_2, \alpha^*(B_1 \times B_2, (C_1 \times C_2))$

(ii) $\alpha^*(A_1 \times A_2,B_1 \times B_2) = \alpha_1(A_1,B_1) \times \alpha_2(A_2,B_2)$

$= \alpha_1(B_1,A_1) \times \alpha_2(B_2,A_2)$

$= \alpha^*(B_1 \times B_2,A_1 \times A_2)$

$\therefore \alpha^*(A_1 \times A_2,B_1 \times B_2) \in T^*$

(iii) Let $A_1 \subseteq C_1$, $A_2 \subseteq C_2$ and $B_1 \subseteq D_1$, $B_2 \subseteq D_2$ then

$\alpha^*(A_1 \times A_2,B_1 \times B_2) = \alpha_1(A_1,B_1) \times \alpha_2(A_2,B_2)$

$\subseteq \alpha_1(C_1,D_1) \times \alpha_2(C_2,D_2)$

$= \alpha^*(C_1 \times C_2, D_1 \times D_2)$

(iv) $\alpha^*(A_0 \times B_0,A_1 \times B_1) = \alpha_1(A_0,A_1) \times \alpha_2(B_0,B_1)$

where A_0, B_0 are the minimal p-open sets of T_1 and T_2 respectively and $A_1 \times B_2 \in P^{X \times Y}$.

$= \alpha_1(A_1, A_0) \times \alpha_2(B_1,B_0)$

$= \alpha^*(A_1 \times B_1, A_0 \times B_0)$

$\therefore \alpha^*$ is a p-a structure on $X \times Y$.

$\therefore (X \times Y, T^*,\alpha^*)$ is a p-a space.

Remarks 3.4. α^* is a called the Product p-a structure and $(X \times Y, T^*, \alpha^*)$ is called the Product p-a space.

Example 3.5. Let (X,T_1,α_1) and (Y,T_2,α_2) be two p-a spaces where (X,T_1) and (Y,T_2) are two p-topological spaces.

Let $\alpha_1(A_1,B_1) = A_1 \cup B_1$ and $\alpha_2(A_2,B_2) = A_2 \cup B_2$ where $A_1,B_1 \in T_1$ and $A_2,B_2 \in T_2$.

Let $T^* = \{A_1 \times A_2 : A_1 \in T_1, A_2 \in T_2 \}$ be the Product p-topology on $X \times Y$.

Let $\alpha^* : P^{X \times Y} \times P^{X \times Y} \to P^{X \times Y}$ be such that

$\alpha^*(A_1 \times A_2,B_1 \times B_2) = \alpha_1(A_1,B_1) \times \alpha_2(A_2,B_2)$

$= (A_1 \cup B_1) \times (A_2 \cup B_2)$

Then $(X \times Y, T^*,\alpha^*)$ is the Product p-a space.

Example 3.6. Let (G_1,T_1,α_1) and $G_2,T_2,\alpha_2)$ be two p-a spaces where G_1 and G_2 are any two groups and T_1, T_2 are usual p-topologies on G_1 and G_2 and α_1,α_2 are usual p-a structures on G_1 and G_2 respectively.

Let $T^* = \{ A_1 \times A_2 : A_1 \in T_1, A_2 \in T_2\}$ be the

Product p-topology on $X \times Y$.

Let $\alpha^* : P^{G_1\times G_2} \times P^{G_1\times G_2} \rightarrow P^{G_1\times G_2}$ be such that

$$\alpha^*(A_1 \times A_2, B_1 \times B_2) = \alpha_1(A_1,B_1) \times \alpha_2(A_2,B_2)$$
$$= (A_1B_1) \times (A_2B_2)$$

Then $(G_1 \times G_2, T^*,\alpha^*)$ is the Product p-a space.

4. SUB p-a SPACE OF PRODUCT p-a SPACE

Proposition 4.1. Let (X,T_1,α_1) and (Y,T_2,α_2) be two p-a spaces where (X,T_1) and (Y,T_2) are p-topological spaces and α_1,α_2 are usual p-a structures on X,Y respectively.

Let $\overline{X} = X \times B_0, B_0$ is the minimal p-open set in T_2

$\overline{T} = \{A \times B_0 : A \in T_1\}$

and $\overline{\alpha} = (A \times B_0, B \times B_0) = \alpha_1(A,B) \times B_0$

Then $(\overline{X},\overline{T}_1,\overline{\alpha}_1)$ is a sub p-a space of $(X \times Y, T^*, \alpha^*)$ where T^* and α^* are the Product p-topology and Product p-a structure respectively on $X \times Y$.

Proof. First we Show that $\overline{T}_1$ is a p-topology on $\overline{X}$.

(i) $\overline{X} = X \times B_0 \in \overline{T}_1$ since $X \in \overline{T}_1$, B_0 is the minimal p-open set in T_2.

(ii) Let A_0 be the minimal p-open set in T_1. Then $A_0 \times B_0$ is the minimal p-open set in $\overline{T}_1$.

(iii) Let $A_1 \times B_0$, $A_2 \times B_0$ be any two elements of $\overline{T}_1$ where $A_1,A_2 \in T_1$

Then $(A_1 \times B_0) \cap (A_2 \times B_0) = (A_1 \cap A_2) \times B_0 \in \overline{T}_1$

since $A_1, A_2 \in T_1 \Rightarrow A_1 \cap A_2 \in T_1$

$\therefore$ $\overline{T}_1$ is a p-topology on $\overline{X}$.

Now we show that $(\overline{X},\overline{T}_1,\overline{\alpha}_1)$ is a sub p-a space of $(X \times Y, T^*, \alpha^*)$.

Let $\overline{\alpha}_1 : P^{\overline{X}} \times P^{\overline{X}} \rightarrow P^{\overline{X}}$ be such that.

(i) $\overline{\alpha}_1(A \times B_0, B \times B_0) = \alpha_1(A,B) \times B_0$, $A,B \in T_1$ and B_0 is the minimal p-open set in T_2.

(ii) $\overline{\alpha}_1(A \times B_0, B \times B_0) = \alpha_1(A,B) \times B_0$

$= \alpha_1(B,A) \times B_0$

$= \overline{\alpha}_1(B \times B_0, A \times B_0)$

$\therefore \overline{\alpha}_1(A \times B_0, B \times B_0) \in \overline{T}_1$

(iii) $\overline{\alpha}_1(A_0 \times B_0, A \times B_0) = \alpha_1(A_0,A) \times B_0$ where A_0 is the minimal p-open set in T_1.

$= \alpha_1(A,A_0) \times B_0$

$= \overline{\alpha}_1(A \times B_0, A_0 \times B_0)$

$\therefore (\overline{X}, \overline{T}_1, \overline{\alpha}_1)$ is a sub p-a space of $(X \times Y, T^*, \alpha^*)$.

Proposition 4.2. Let (X,T_1,α_1) and (Y,T_2,α_2) be two p-a spaces where (X, T_1) and (Y,T_2) are p-topological spaces, α_1 and α_2 are p-a structures on X and Y respectively. Let $(\overline{X}, \overline{T}_1, \overline{\alpha}_1)$ be a sub p-a space of $(X \times Y, T^*, \alpha^*)$ where T^* and α^* are the product p-topology and Product p-a structure respectively on $X \times Y$.

Let $f : (X,T_1,\alpha_1) \rightarrow (\overline{X}, \overline{T}_1, \overline{\alpha}_1)$ be an onto mapping such that

$f(\{x\}) = \{x\} \times B_0$ where B_0 is the minimal p-open set in T_2

Then f is a p-a homomorphism.

Proof. (i) $f(x) = X \times B_0 \in \overline{T}_1$ whenever $X \in T_1$

$\therefore$ f is p-open.

$f^{-1}(\overline{X}) = f^{-1}(X \times B_0) = X \in T_1$ whenever $\overline{X} \in \overline{T}_1$

$\therefore$ f is p-continuous.

$\therefore$ f is both p-open and p-continuous.

(ii) $f(\alpha_1(A,B)) = \alpha_1(A,B) \times B_0$, $A,B \in T_1$

$= \overline{\alpha}_1(A \times B_0, B \times B_0)$

$= \overline{\alpha}_1(f(A), f(B))$

and $\alpha_1(f^{-1}(\overline{A}),f^{-1}(\overline{B})) = \alpha_1(f^{-1}(A\times B_0),f^{-1}(B\times B_0))$

$= \alpha_1(A,B)$

$= f^{-1}(\alpha_1(A,B) \times B_0)$

$= f^{-1}(\overline{\alpha}_1(A \times B_0, B \times B_0))$

$= f^{-1}(\overline{\alpha}_1(\overline{A},\overline{B}))$

$\therefore$ f is a p-a homomorphism.

Remarks 4.3. $(\overline{X},\overline{T}_1,\overline{\alpha}_1)$ may be identified by (X,T_1,α_1).

(X,T_1,α_1) is called the projection of $(X \times Y, T^*, \alpha^*)$ by (Y,T_2,α_2)

Similarly we may speak about the projection

(Y,T_2,α_2) of $(X \times Y, T^*, \alpha^*)$ by (X,T_1,α_1).

ACKNOWLEDGEMENT

I offer my heartiest thanks to Prof. Dr. N. R. Das, Head of the Department of Mathematics, Gauhati University, for his valuable advices in preparing this paper.

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