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$\mathcal{H}^*$ RELATION ON A π-INVERSE SEMIGROUP *

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ABSTRACT : The aim of this paper is to give a necessary and sufficient condition for the $\mathcal{H}^*$ relation to be a congruence on a π-inverse semigroup. The work of M. K. Sen on an inverse semigroup is further developed.

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Key words : π-inverse semigroup, GV-inverse semigroup, $\mathcal{H}^*$ relation, Congruence.

1. INTRODUCTION

In [1], generalized Green's relations $\mathcal{L}^*, \mathcal{R}^*, \mathcal{H}^*, \mathcal{T}^*$ on π-regular semigroups are defined. It is noted that $\mathcal{H}^*$ relation is a congruence on quasi-clifford semigroups and GV-inverse semigroups. However, $\mathcal{H}^*$ relation is not necessary a congruence on π-orthodox semigroups, π-inverse semigroups, strong π-inverse semigroups, and even GV-semigroups. We will give a necessary and sufficient condition for the $\mathcal{H}^*$ relation to be a congruence on π-inverse semigroups. As a corollary, the result of [3] is given.

An element a of semigroup S is called π-regular if there exists a positive integer m such that $a^m \in a^mSa^m$. The semigroup is called π-regular if all its elements are π-regular. In particular, if a π-regular semigroup contains only one idemodent, then it is called a π-group. By a π-inverse semigroup, we mean a π-regular semigroup in which every regular element has an unique inverse element. Call a π-inverse semigroup stong if its set of idempotents forms a subsemigroup. A semigroup is called a GV-semigroup if it is π-regular and all its regular elements are completely regular. Call a GV-semigroup GV-inverse if it is π-inverse. By a quasi-clifford semigroup, we mean a GV-inverse semigroup whose set of idempotents forms a semilattice.

Throughtout this paper, E(S) is the set of all idempotents of the semigroup S, and Reg S is the set of all regular elements of the semigroup S. Also, we write $a^{\prime\prime} = r(a)$, where

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a is an arbitrary element of the π-regular semigroup S and n is the smallest positive integer such that $a^n \in Reg\ S$. For the sake of convenience, we just call n the smallest regular power of a.

In [1], *-Green's relations on the π-regular semigroup S are introduced :

$$a\mathcal{L}^*b \Leftrightarrow Sr(a) = Sr(b)\ ;$$

$$a\mathcal{R}^*b \Leftrightarrow r(a)S = r(b)S\ ;$$

$$a\mathcal{T}^*b \Leftrightarrow Sr(a)S = Sr(b)S\ ;$$

$$\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*.$$

Throughout this paper, $\mathcal{L}^*$, $\mathcal{R}^*$ *and* $\mathcal{H}^*$ are *-Green's relations on the semigroup S. For other notations and terminologies not given in this paper, the reader is refered to the texts of J. M. Howie [2] and Bogdanovic [1].

2. PRELIMINARIES

We give here some descriptions for the π-inverse semigroups, the GV-inverse semigroups and the $\mathcal{H}^*$ relations on the π-inverse semigroups.

Lemma 2.1. [1] Let S be a π-regular semigroup. Then

1. H_e^* is a π-group, for all $e \in E(S)$, where H_e^* is the $\mathcal{H}^*$-class containing e of S ;
2. a$\mathcal{H}^*$b if and only if there exists $a' \in V(r(a))$ and $b' \in V(r(b))$ such that $r(a)a' = r(b)b'$ and $a'r(a) = b'r(b)$, for all $a, b \in S$;
3. If S is a π-group, then the group Reg$S$ is an ideal of S ;
4. S is π-inverse if and only if there exists a positive integer n such that $(ef)^n = (fe)^n \in E(S)$, for all $e, f \in E(S)$;
5. S is strong π-inverse if and only if $E(S)$ is a semilattice ; if and only if Reg$S$ is an inverse subsemigroup ;
6. S is a quasi-clifford semigroup if and only if Reg$S$ is a clifford subsemigroup of S.

Lemma 2.2. [1] Let S be a π-regular semigroup. Then the following conditions are equivalent :

1. S is GV-inverse ;
2. S is a semilattice of π-groups ;

3. S is a GV-semigroup and there exists a positive integer n such that $(ef)^n = (fe)^n$, for all $e, f\ E(S)$;

4. $\mathcal{H}^* = \mathcal{T}^*$ is a semilattice congruence on S.

Lemma 2.3. Let S be a π-inverse semigroup and $a\mathcal{H}^*b$ where $a, b \in$ S. Then $r(a)r(b)^{-1}$, $r(b)r(a)^{-1}$, $r(a)^{-1}r(b)$, and $r(b)^{-1}r(a) \in \bigcup_{e\in E(s)} H_e^*$

Proof. Let n be the smallest regular power of $r(a)r(b)^{-1}$. Then, by the definition of $\mathcal{H}^*$, we have $r(a)r(b)^{-1}\mathcal{H}^*(r(a)r(b)^{-1})^n$.We show here $(r(a)r(b)^{-1})^n\mathcal{H}^*r(b)r(b)^{-1}$. Noting that S is a π-inverse semigroup and using (2) of Lemma 2.1, we can show

$$
\begin{aligned}
Sr(a)r(b)^{-1} &= Sr(b)r(b)^{-1}r(b)r(b)^{-1} \\
&= Sr(b)r(a)^{-1}r(a)r(b)^{-1} \\
&\subseteq Sr(a)r(b)^{-1} \\
&= Sr(a)r(a)^{-1}r(a)r(b)^{-1} \\
&= Sr(b)r(b)^{-1}r(a)r(b)^{-1} \\
&= Sr(b)r(b)^{-1}r(b)r(b)^{-1}r(a)r(b)^{-1} \\
&= Sr(b)r(a)^{-1}r(a)r(b)^{-1}r(a)r(b)^{-1} \\
&= Sr(b)r(a)^{-1}(r(a)r(b)^{-1})^2 \\
&= Sr(b)r(a)^{-1}r(b)r(b)^{-1}(r(a)r(b)^{-1})^2 \\
&\subseteq Sr(b)r(b)^{-1}(r(a)r(b)^{-1})^2 \\
&= Sr(b)r(a)^{-1}(r(a)r(b)^{-1})^3 \\
&\subseteq Sr(b)r(b)^{-1}(r(a)r(b)^{-1})^3 \\
&\subseteq \cdots \\
&\subseteq S(r(a)r(b)^{-1})^n.
\end{aligned}
$$

Also,

$$
\begin{aligned}
S(r(a)r(b)^{-1})^n &= S(r(a)r(b)^n r(b)r(b)^{-1} \\
&\subseteq Sr(b)r(b)^{-1}.
\end{aligned}
$$

Hence,

$$(r(a)r(b)^{-1})^n \mathcal{L}^*r(b)r(b)^{-1}.$$

Similarly,

$$(r(a)r(b)^{-1})^n \mathcal{R}^*r(b)r(b)^{-1}.$$

As a consequence, we derive that

$$(r(a)r(b)^{-1})\,\mathcal{H}^*(r(a)r(b)^{-1}\mathcal{H}^*r(b)r(b)^{-1}.$$

This leads to $r(a)r(b)^{-1} \in \bigcup_{e\in E(S)} H_e^*$. Similarly, we can show that $r(b)r(a)^{-1}, r(a)^{-1}r(b)$ and $r(b)^{-1}r(a) \in \bigcup_{e\in E(S)} H_e^*$. The proof is completed.

Lemma 2.4. Let S be a π-inverse semigroup and $\mathcal{H}^*$ is a congruence on S. Then $\bigcup_{e\in E(S)} H_e^*$ is a GV-inverse subsemigroup of S.

Proof. Suppose that $a \in H_e^*$ and $b \in H_f^*$ where $e, f \in E(S)$. Let n be a positive integer such that $(ef)^n = (fe)^n \in E(S)$. Because $\mathcal{H}^*$ is a congruence on S, we have $ab\mathcal{H}^*ef$ and $(ab)^n\mathcal{H}^*(ef)^n = (fe)^n\mathcal{H}^*(ba)^n$. If we let m be the smallest regular power of ab and k be a positive integer such that $m^k \geq n$, then

$$ab\mathcal{H}^*(ab)^m\mathcal{H}^*(ab)^{m^2}\mathcal{H}^*\ldots\mathcal{H}^*(ab)^{m^k}\mathcal{H}^*(ef)^{m^k}\mathcal{H}^*(ef)^n.$$

Hence $ab \in \bigcup_{g\in E(S)} H_g^*$ since $(ef)^n \in E(S)$. Thus, by $a\mathcal{H}^*r(a)$, $\bigcup_{e\in E(S)} H_e^*$ is a π-regular subsemigroup of S. Similarly, $ba\mathcal{H}^*(fe)^n$. Consequently, $ab\mathcal{H}^*ba$. This means that $\mathcal{H}^*$ is a semilattice congruence on $\bigcup_{e\in E(S)} H_e^*$. Hence, by Lemma 2.1 and Lemma 2.2, we finally obtain $\bigcup_{e\in E(S)} H_e^*$ is a GV-inverse subsemigroup of S.

3. MAIN RESULTS

Theorem 3.1. Let S be a π-inverse semigroup. Then $\mathcal{H}^*$ relation is a congruence on S if and only if $T = \bigcup_{e\in E(S)} H_e^*$ is a GV-inverse subsemigroup of S and $r(ab)\mathcal{H}^*r(a)r(b)$ for all $a,b \in S$

Proof. By the above lemmas, we only need to prove the sufficiency.

Suppose now T is a GV-inverse subsemigroup of S and $r(ab)\mathcal{H}^*r(a)r(b)$ for all $a,b \in S$. Let $a\mathcal{H}^*b$, then by Lemma 2.3, we have $r(a)r(b)^{-1}, r(b)r(a)^{-1}$, $r(a)^{-1}r(b)$ *and* $r(b)r(a)^{-1} \in T$. If we write $\mathcal{H}^{*T}$ for the $\mathcal{H}^*$ relation on T, then since T is GV-inverse semigroup, $\mathcal{H}^{*T}$ is a semilattice congruence on T. Suppose that $c \in S$, using the above facts, we immediately have

$r(c)^{-1}r(c)r(a)r(b)^{-1}\mathcal{H}^{*^T}r(a)r(b)^{-1}r(c)^{-1}r(c)$.

$\mathcal{H}^{*^T}r(a)r(b)^{-1}r(b)r(b)^{-1}r(c)^{-1}r(c)\mathcal{H}^{*^T}r(a)r(b)^{-1}r(c)^{-1}r(c)r(b)r(b)^{-1}$

Hence, there exists $x \in T$ such that $r(c)^{-1}r(c)r(a)r(b)^{-1} = xr(a)r(b)^{-1}r(c)^{-1}r(c)r(b)r(b)^{-1}$. By noting that $r(b)^{-1}r(b) = r(a)^{-1}r(a)$, we can show

$r(c)r(b) = r(c)r(c)^{-1}r(c)r(a)r(b)^{-1}r(b) = [r(c)xr(a)r(b)^{-1}r(c)^{-1}]r(c)r(b)$

Similarly, there exists $y \in T$ such that

$r(c)r(b) = [r(c)yr(a)r(b)r(a)^{-1}]r(c)r(a)$.

Consequently, we derive that $r(c)r(a)\mathcal{L}^*r(c)r(b)$. Also it is obvious that $r(c)r(a)\mathcal{R}^*r(c)r(b)$.

Hence $r(c)r(a)\mathcal{H}^*r(c)r(b)$. Thereby, $ca\mathcal{H}^*r(ca)\mathcal{H}^*r(c)r(a)\mathcal{H}^*r(c)r(b)\mathcal{H}^*r(cb)\mathcal{H}^*cb$. This shows the $\mathcal{H}^*$ relation on S is a left congruence on S. In the same way, we can obtain $ac\ \mathcal{H}^*bc$, that is $\mathcal{H}^*$ is right compatible. Thus, $\mathcal{H}^*$ is indeed a congruence on S. The proof is completed.

Corollary 3.2. Let S be a strong π-inverse semigroup. Then $\mathcal{H}^*$ is a congruence on S if and only if $T = \bigcup_{e \in E(s)} H_e^*$ is a quasi-clifford subsemigroup of S and $r(ab)\mathcal{H}^*r(a)r(b)$ for all $a,b \in S$.

Proof. By noting that a GV-inverse semigroup whose set of all idempotents forms a semilattice is a quasi-clifford semigroup, the proof is completed.

Corollary 3.3 [3] Let S be a inverse semigroup, then $\mathcal{H}^*$ is a congruence on S if and only if $\bigcup_{e \in E(s)} H_e^*$ is a clifford subsemigroup of S.

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