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A NOTE ON $\mathcal{H}$ RELATION ON AN INVERSE SEMIGROUP

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ABSTRACT. *$\mathcal{H}$ relation is studied on an inverse semigroup and it is established that an inverse semigroup is cryptic if and only if it is group closed.*

$\mathcal{H}$ relation on an inverse semigroup S is not in general a congruence relation. If S is a group bound inverse semigroup, then also $\mathcal{H}$ relation may not be a congruence relation. In this short note we want to show that an inverse semigroup is cryptic [1] (ie, $\mathcal{H}$ is a congruence relation) if and only if it is group closed (i.e, $G_r(S)$ is a subsemigroup of S).

Definition 1. [2] A semigroup S is said to be group bound if some power of each element of S lies in a subgroup of S. An element a of a semigroup S is said to be a group element if a lies in a subgroup of S. We give an example of an inverse semigroup which is group bound but not cryptic.

Example 2. Let X = { 1, 2, 3 }, and $\mathcal{I}$ (X) be the symmetric inverse semigroup on X. Since $\mathcal{I}$ (X) is finite, $\mathcal{I}$ (X) is group bound. Since

$\left\{\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix},\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}\right\}$ is subgroup of $\mathcal{I}$ (X). $\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$ is a group element. But

$$\begin{pmatrix}1&2\\1&2\end{pmatrix}\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}=\begin{pmatrix}1&2\\1&3\end{pmatrix},\begin{pmatrix}1&2\\1&2\end{pmatrix}\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}=\begin{pmatrix}1&2\\1&2\end{pmatrix}$$

Hence $\begin{pmatrix}1&2\\1&3\end{pmatrix}$ is not in the same $\mathcal{H}$ class as $\begin{pmatrix}1&2\\1&2\end{pmatrix}$. Consequently $\mathcal{H}$ is not a congruence relation on S.

Next we give an example of group bound inverse semigroup which is not a Clifford semigroup but $\mathcal{H}$ is a congruence relation.

Example 3. Let S = { a, b, c, d, o }. Define a binary operation on S by the following table :

	a	b	c	d	o
a	o	d	a	o	o
b	c	o	o	b	o
c	o	b	c	o	o
d	a	o	o	d	o
o	o	o	o	o	o

It can be checked easily that (S,.) is a group bound inverse semigroup on which $\mathcal{H}$ relation is a congruence relation but not a Clifford semigroup.

Theorem 4. Let S be an inverse semigroup. The following conditions are equivalent on S :

(i) $\mathcal{H}$ is a congruece relation on S.

(ii) $G_r(S)$ (the set of all group elements of S is a Clifford subsemigroup of S.

(iii) $(\forall e \in E(S), \forall a \in G_r(S)$ $ea, ae \in G_r(S)$.

(iv) $(\forall e \in E(S), \forall a \in G_r(S)$ $ea = ae$, i.e, $E(S) \subseteq C(G_r(S))$.

Proof. (i)⇒(ii). Let $a, b \in G_r(S)$. There exist idempotents e, f of S such that $a \in H_e$ and $b \in H_f$. Since $\mathcal{H}$ is a congruence relation, $ab \in H_{ef}$. So, $G_r(S)$ is a subsemigroup of S. Let $e \in E(S)$, $a \in G_r(S)$. We show that $ea = ae$. Assume that $a \in H_f$ for some $f \in E(S)$. Then $ea \,\mathcal{H}\, ef$ and $ae \,\mathcal{H}\, fe$. Hence $ea \,\mathcal{H}\, ae$ which implies that $eaa^{-1} \,\mathcal{H}\, aea^{-1}$, where a^{-1} is the group inverse of a. Now $eaa^{-1}, aea^{-1} \in E(S)$. Consequently, $eaa^{-1} = aea^{-1}$ and this shows that $ea = (eaa^{-1})a = (aea^{-1})a = aa^{-1}ae = ae$.

(ii)⇒(iii). It is obvious.

(iii)⇒(iv). Suppose that $ea \in H_f$ and $ae \in H_g$., where $f, g \in E(S)$. There exists $x \in H_f$ such that $eax = f$. Then $fe = ef = e(eax) = eax = f$. Similarly, $eg = ge = g$. Now $eae = (ea)e = f(ea)e = f(ae)$, $eae = e(ae) = eg(ae) = g(ae)$. Let $(ae)^{-1}$ be the group inverse of ae. Then $f(ae)(ae)^{-1} = g(ae)(ae)^{-1}$, i.e, $fg = g$. Similarly we can show that $fg = f$. Hence $f = g$. Now, $ea = (ea)f = (eae)f = e(ae)f = eae$, and $ae = f(ae) = f(eae) = f(ea)e = eae$. Hence $ea = ae$.

(iv)⇒(i). Assume that $a \,\mathcal{H}\, b$. There exist $x, y \in H_{a^{-1}a}$, $u, v \in H_{aa^{-1}}$ such that $a = bx$, $b = ay$, $a = ub$ and $b = va$. Now, $ca = cub = cc^{-1}cub = cuc^{-1}cb$. This implies that $ca \in Scb$. Again, $cb = cc^{-1}cb = c(c^{-1}c)va = cvc^{-1}ca$, which shows that $cb \in Sca$. Hence $ca \,\mathcal{L}\, cb$. Obviously, $ca \,\mathfrak{R}\, cb$. Thus it follows that $ca \,\mathcal{H}\, cb$. Similarly we can show that $ac \,\mathcal{H}\, bc$. Therefore, $\mathcal{H}$ Is a congruence relation.

Definition 5. A semigroup S is said to be quasisemilattice if it is inverse, perodic and $\mathcal{H} = 1_s$.

Theorem 6. An inverse semigroup S is cryptic and group bound if and only if it is a quasi semilattice of groups (i.e, there exists a congruence ρ on S such that S/ρ is a quasisemilattice and each eρ is a group for every $e \in E(s)$).

Proof. ($\Rightarrow$) This follows immediately.

($\Leftarrow$). Suppose that there exists a congruence ρ such that S/ρ is a quasi semilattice and each $e\rho$ is a group for every $e \in E(S)$. Let $a \in S$. By the hypothesis, there exists a natural number n such that $(a\rho)^n$ is an idempotent. By Lallement's Lemma, we have that $a^n\rho = e\rho$ for some $e \in E(S)$. Since $e\rho$ is a group, a^n lies in a subgroup of S. So S is group bound. Obviously ρ is idempotent seperating, and hence $\rho \subseteq \mathcal{H}$. Let $(a,b) \in \mathcal{H}$. Then $a\rho\, \mathcal{H}\, b\rho$. Since S/ρ is a quasisemilattice, the relation $\mathcal{H}$ is the identity relation on S/ρ. Hence $a\rho = b\rho$, showing that $(a,b) \in \rho$. Thus it follows that $\mathcal{H} = \rho$. Consequently, $\mathcal{H}$ is a congruence relation on S.

REFERENCES

1. N.R. Reilly, Minimal non-cryptic varieties of inverse semigroups, Quart. J. Math. Oxford (2) 36 (1985) 467-487.
2. T.E. Hall and W. D. Munn, Semigroups satisfying minimal conditions II, Glasgow Math. J. 20 (1979), 133-140.

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