Jour. Pure Math., Vol. 14, 1997, pp. 10-17

EXISTENCE AND UNIQUENESS OF AN INVARIANT INTEGRAL ON A MORE GENERAL CLASS OF FUNCTIONS IN A GENERALIZED TOPOLOGICAL GROUP

S. Ganguly, C. K. Basu and S. Sinha Roy

ABSTRACT : In this note, we initiate θ-Harr integral on a more general class of functions than the class of real valued continuous functions with compact support, in a locally θ-H-closed [6], θ-topological group [4]. In this space, which is more general than locally compact topological group, we establish the existence and essential uniqueness of θ-Haar integral.

1995 AMS Subject Classification : 28C 99

Key Words and Phrases : H-set, θ-open, Urysohn, locally-θ-H-closed, θ-CR, θ-T_2, θ-Haar integral, continuous function with H-set support.

1. INTRODUCTION

Veličko, [7] initiated the concepts of H-sets and θ-open sets and Fomin [2] introduced weaker forms of continuity under the terminology θ-continuity. Using such concepts the notion of locally θ-H-closedness [6] and θ-topological group [4] were introduced. It can be noted that every compact set is an H-set but not conversely; so it is natural that the class $\mathcal{H}$ of all real valued continuous functions with H-set support is larger than the class of all real valued continuous functions with compact support. So our objective would be to take these general class of functions under the purview of integration theory. In this paper we have shown the existence of an invariant integral on $\mathcal{H}$ and that too in a generalized topological group.

2. DEFINITIONS AND PREREQUISITES

For the definition of H-sets, θ-poen sets, θ-continuous function, θ-homeomorphism and Urysohn space we refer to the book of Porter and Woods [5]. A topological space X is said to be locally θ-H--closed [6] if every point x of X has a θ-open set containing x whose closure is an H-set. A topological space X is said to be θ-completely regular

[6] (θ-CR, for short) if for any point p and a θ-closed set H in X with $p \notin H$, there exists a continuous function $f : X \to [0, 1]$ such that $f(p) = 0$ and $f(H) = 1$; X is called θ-T_2 [6] if any two distinct points can be separated by θ-open sets. Clearly every θ-T_2 space is Urysohn and H-set in a Urysohn space is θ-closed [3]. In [3] it was established that there exists a space X which is locally θ-H-closed, θ-CR, θ-T_2 but non-regular, non-locally compact, non-H-closed. A group X with multiplication map m and inversion map i, endowed with topology τ, is called a θ-topological group [4] if m and i are θ-continuous and each left as well as each right translation is an open map. Clearly every topological group is a θ-topological group but not conversely [4]. A real valued function f on X is said to have H-set support if there exists an H-set K of X outside of which f vanishes.

3. EXISTENCE OF θ-HAAR INTEGRAL

Let X be a locally θ-H-closed θ-T_2, θ-CR, θ-topological group, $\mathcal{H}$ be collection of all real valued continuous functions on X with H-set support and $\mathcal{H}^+$ be the subclass of $\mathcal{H}$ consisting of all $f \in \mathcal{H}$ for which $f \geq 0$ and f is not identically zero i.e.

$$\mathcal{H}^+ = \{f \in \mathcal{H} : f \geq 0 \text{ and } f \neq 0\}$$

It can be easily shown that for any $a \in x$, the function $x \to a^{-1}x$ is θ-continuous and infact θ-homeomorphism. Further for any $a \in x$ and for any real valued function f on X, the function $f_a(x) = f(a^{-1}x)$ is θ-continuous iff f is θ-continuous. Also f_a is a function having H-set support iff f is so.

Definition 3.1. A θ-Haar integral I for X is a +ve linear form on $\mathcal{H}$ not identically zero such that $I(f_a) = I(f)$ for all $f \in \mathcal{H}$ and for all $a \in X$.

Our goal is to show the existence of an non-zero linear functional I on $\mathcal{H}^+$ s.t. $I(f_a) = I(f)$ for all $f \in \mathcal{H}^+$ and then extend the funtional to the whole of $\mathcal{H}$ as follows:

Let $f \in \mathcal{H}$ then f^+ and $f^- \in \mathcal{H}^+$ and hence $I(f^+)$ and $I(f^-)$ exist. Define the extension $I(f) = I(f^+) - I(f^-)$.

The following lemma is the basis for showing the existence of θ-Haar Integral.

Lemma 3.2. For f, $h \in \mathcal{H}^+$, there exist $a_1, \ldots, a_n$ of X and $c_i > 0$, $i = 1, 2, \ldots,$ n such that $f \leq \sum_{i=1}^{n} c_i h_{a_i}$

Proof. Since $h \in \mathcal{H}^+$, h is not identically zero on X and hence for some $x \in X$, $h(x) > 0$, then the set $V = \{x \in X : h(x) > 0\}$ is non-empty θ-open set as $x \in X$ and h is θ-continuous. Again X being locally-θ-H-closed, there exists a θ-open set U such that $x \in U \subset \overline{U} \subset V$ and is an H-Set ([6]). Now h being θ-continuous, $h(\overline{U})$ is an H-set in R, and hence a compact subset of R So h is bounded on $\overline{U}$

and attains its bounds on $\overline{U}$ and hence there exists $\eta > 0$ (where η = g. 1.b. of h on $\overline{U}$) such that h (x) $\geq \eta$ on U.

Now f being in $\mathcal{H}^+$, there exist an H-set K such that f = 0 on X – K. The set {aU : a $\in$ X} is an θ-open covering of the H-set K (each translation being θ-homorphism). So there exist $a_1, \ldots, a_n$ such that $K \subset \bigcup_{i=1}^{n} a_i U$. If we take M = 1.u.b. of f, $c_i = \frac{M}{\eta}$ for i = i = i, 2, ... , n, then $f(x) \leq \sum_{i=1}^{n} c_i h_{a_i}(x)$.

Definition 3.3. For f, g $\in \mathcal{H}^+$ we define

$$(f : g) = \text{g.1.b} \{ \sum_{i=1}^{n} c_i : \; f(x) \leq \sum_{i=1}^{n} c_i g_{a_i}(x). \; c_i > 0\}$$

in view of the above lemma, for any two functions f, g $\in \mathcal{H}^+$, (f : g) exists and is finite. Also (f : g) > 0 and $(f : g) \geq \frac{M}{m}$ where M = 1.u.b of f and m = 1.u.b. of g. Further it satisfies the following properties.

(i) $(f_a : g) = (f : g)$ for any a $\in$ x

(ii) $(cf : g) = c\,(f : g)$ for $c \geq 0$

(iii) $(f + g : h) \leq (f : h) + (g : h)$, $\forall f, g, h \in \mathcal{H}^+$

(iv) $f \leq g \Rightarrow (f : h) \leq (g : h)$

(v) $(f : h) \leq (f : g)\,(g : h)$

(vi) $\frac{1}{(h:f)} \leq \frac{(f:g)}{(h:g)} \leq (f : h)$

Let $f_0 \in \mathcal{H}^+$ be fixed and for any f, g $\in \mathcal{H}^+$, we define $I_g(f) = \frac{(f:g)}{(f_0:g)}$. Since $(f : g) \leq (f : f_0)\,(f_0 : g)$ it follows that $0 \leq I_g(f) \leq (f : f_0)$ for every f $\in \mathcal{H}^+$. Let X_f = [o, : (f, f_0)]. Let $Y = \pi_{f \in \mathcal{H}^+} X_f$. Clearly Y is compact. For each θ-open set V containing e, let F_V = cl {$I_g \in$ Y : Support of g is in V}. Such a g always exists as X is locally θ-H-closed and θ-CR. So F_V is non-empty. T}. he family {F_V : V is θ-open set containing e} has finite intersection property and Y being compect,

$\cap$ {F_V : V is θ-open set containing e} $\neq \phi$. Let I = (I (f)), where I (f) is the f-th co-ordinate of I belongs to the intersection. We shall show that I is the required functional on $\mathcal{H}^+$.

Since I = (I (f)) $\in \cap$ {F_V : V is θ-open set cintaining e} so for any $\varepsilon > 0$ and $f_1, f_2, \ldots, f_n \in \mathcal{H}^+$, the basic open set $(I(f_1) - \varepsilon, I(f_1) + \varepsilon) \times \ldots \times (I(f_n) - \varepsilon, I(f_n) + \varepsilon)$

$$\times \underset{f \neq f_1,\ldots,f_n}{\pi X_f}$$

Contains a point $(I_g (f)) \in \{I_g \in Y$: Support of g is in V}. Hence for i = 1, 2, ... n, $| I_g (f_i) - I(f_i) | < \varepsilon$. .. .. (B)

Since the following properties (i)— (iv) below hold for $I_0 = I_g$, it can easily be obtained (i)— (iv) for I_0 = I.

(i) $I_0 \neq 0$, $I_0 (f) > 0$ if $f \neq 0$

(ii) $I_0 (f_a) = I_0 (f)$

(iii) $I_0 (cf) = cI_0 (f)$, $c \geq 0$

(iv) $I_0 (f_1 + f_2) \leq I_0 (f_1) + I_0 (f_2)$.

In order to show I is left invariant non-zero +ve linear functional on $\mathcal{H}^+$ it is sufficient to show that the reverse inequality in (iv) holds for I_0 = I.

For this we need two Propositions :–

Proposition 3.4. If $f \in \mathcal{H}$, then given any $\varepsilon > 0$, there exists a θ-open set U containing e such that $| f(z) - f(g) | < \varepsilon$ wherever $gz^{-1} \in U$.

Proof. We define $U^* = \{y \in x : | f(yz) - f(z) | < \varepsilon$ for all $z \in X\}$. Since $f \in \mathcal{H}$, there exists an H-set K such that f = 0 on X – K. As X is locally θ-H-closed, there exists an θ-open set L containing e such that $\overline{L}$ is an H-set Now the inversion map being θ-homeomorphism, L^{-1} is θ-open sat containing e as well as $(\overline{L})^{-1}$ is an H-set (as θ-continuous mapping carries H-set into an H-set). So $V = L \cap L^{-1} \subset \overline{L} \cap (\overline{L})^{-1}$. X being an θ-T_2 and hence Urysohn, the H-set $\overline{L} \cap (\overline{L})^{-1}$ is θ-closed. $\overline{V}$ being closure of an θ-open set implies $\overline{V}$ = θ-closure of V and as X is Urysohn, θ-closure of V is θ-closed set. So $\overline{V}$ is a θ-closed set contained in a H-set $\overline{L} \cap (\overline{L})^{-1}$. So $\overline{V}$ is H-set. Furthermore as the multiplication map is θ-continuous and $K \times \overline{V}$ is and H-set, then $K.\overline{V}$ is H-set. Clearly $K \subset K\overline{V}$. Given any $g \in K\overline{V}$, f being continuous and hence θ-continuous, there exist θ-open sets V_g containing g and U_g containing e such that $| f(yz) - f(z) | < \varepsilon$ wherever $y \in U_g$ and $z \in V_g$. Since $K\overline{V}$, is an H-set, there exist $g_1, g_2, \ldots, g_n \in K\overline{V}$ such that $K\overline{V} \subset \bigcup_{i=1}^{n} v_{g_i}$. Define $U = U_{g_i} \cap \ldots \cap U_{g_i} \cap V$.

Then U is an θ-open set containing e. We shall show that $U \subset U^*$. Let $Y \in U$ and $z \in X$. If $z \in K\bar{V}$ then $z \in V_{g_i}$ for some i. Since $Y \in U_{g_i}$ it follows that $|f(yz) - f(z)| < \varepsilon$. So $Y \in U^*$. On the otherhand if $z \notin K\bar{V}$, then $z \notin K$, hence f (z) = 0 ; since $yz \notin K$ ($yz \in K$ would imply $z \in Ky^{-1} \subset KV \subset K\bar{V}$,— a contradiction), f (yz) = 0. So $|f(yz) - f(z)| = 0 < \varepsilon$. So $y \in U^*$. Therefore $U \subset U^*$ and hence the proposition.

Proposition 3·5. Let $f_1, f_2 \in \mathcal{H}^+$ and $\varepsilon > 0$. Then there exists a θ-open set U containing e such that for all $g \in \mathcal{H}^+$ having support in U, $I_g(f_1) + I_g(f_2) \leq I_g(f_1 + f_2) + \varepsilon$.

Proof. Clearly $f_1 + f_2 \in \mathcal{H}^+$. Let K be the H-set s. t. $f_1 + f_2$ vanishes outside K. Again X being locally θ-H-closed and θ-CR, there exists $f \in \mathcal{H}^+$ such that $f'(K) = 1$. Let δ_1 be arbitrary positive number and let $f = f_1 + f_2 + \delta_1 f'$. Clearly $f \in \mathcal{H}^+$. We define two functions

$$h_i(x) = \frac{f_i(x)}{f(x)} \text{ if } f(x) \neq 0$$

$$= 0 \text{ otherwise}$$

for i = 1, 2.

Since $h_i(x) \in \mathcal{H}^+$ given any $\delta_2 > 0$ by Proposition 3.4, there exist a θ-open set U containing e s. t.

$$|h_i(z) - h_i(a)| < \delta_2$$

whenever $az^{-1} \in U$ and $a, z \in X$.

Let $g \in \mathcal{H}^+$ be such that its support lies in U and we consider $f(x) \leq \sum_i c_i\, g(z_i^{-1} z)$. If $z_i^{-1} z$ is such that $g(z_i^{-1} z) \neq 0$ then $z_i^{-1} z \in U$ and therefore,

$$|h_i(z) - h_i(z_i)| < \delta_2$$

But $f_i(z) = f(z)\, h_i(z) \leq \sum_i c_i\, g(z_i^{-1} z)\, h_i(z)$

$\leq \sum_i c_i\, g(z_i^{-1} z)(h_i(z_i) + \delta_2)$. Therefore

$(f_i : g) \leq \sum_i c_i (h_i(z_i) + \delta_2)$

Hence $(f_1 : g) + (f_2 : g) \leq \sum_i c_i (h_1(z_i) + h_2(z_i) + 2\delta_2) \leq \sum C_i (1 + 2\delta_2)$ (as $h_i(z_i) + h_2(z_i) \leq 1$)

This implies, $(f_1 : g) + (f_2 : g) \leq (f : g)(1 + 2\delta_2)$

For a fixed non-zero $f_o \in \mathcal{H}^+$ if we divide the above inequality by $(f_o : g)$ then $I_g(f_1) + I_g(f_2) \leq I_g(f)(1 + 2\, d_2)$ ... ... (A) Also it is easy to show that $I_g(f) \leq I_g(f_1) + I_g(f_2) + \delta_1 I_g(f')$. Therefore $I_g(f_1) + I_g(f_2) \leq (1 + 2\delta_2)(I_g(f_1) + I_g(f_2) + \delta_1 I_g(f'))$. Since δ_1 are arbitraty we can choose δ_1 and δ_2 in such a manner such that $\{2\delta_2 (I_g(f_1) + I_g(f_2) + \delta_1 I_g(f')) + \delta_1 I_g(f')\} < \varepsilon$. So $I_g(f_1) + I_g(f_2) \leq I_g(f_1 + f_2) + \varepsilon$ (from relation B) Hence the proposition.

For a given $\varepsilon > 0$ from relation (B) (before Proposition 3.4), we have,
$|I(f_1 + f_2) - I_g(f_1 + f_2)| < \varepsilon$.
By Proposition 3.5, $I_g(f_1) + I_g(f_2) \leq I_g(f_1 + f_2) + \varepsilon$.
Therefore $I(f_1) + I(f_2) \leq I_g(f_1) + I_g(f_2) + 2\varepsilon$
$\leq I_g(f_1 + f_2) + 3\varepsilon \quad \leq I(f_1 + f_2) + 4\varepsilon$.
Since ε is arbitrary, we have
$I(f_1) + I(f_2) \leq I(f_1 + f_2)$.
So I is additive.

Therefore I can be extended to $\mathcal{H}$ and that extended I will be our required θ-Haar Integral.

Hence we get a theorem.

Theorem 3·6. Let X be a locally θ-H-closed θ-T_2, θ-CR, θ-topological group. Then there exists a non-trivial θ-Haar integral on $\mathcal{H}$.

4. UNIQUENESS OF θ-HAAR INTEGRAL

Suppose J is any other θ-Haar integral for X. We shall show that J = CI for a suitable constant $C > 0$. The following lemma is important and as the proof is quite similar to the proof of Lemma 15 Art. 76 [1]. we shall omit the Proof.

Lemma 4.1. If K is an H-set of X, and f, f' $\in \mathcal{H}^+$ such that support of f lies in K and f' = 1 on K then given any $\varepsilon > 0$, there exists a θ-open set U containing e such that

$$(f : g) \leq \varepsilon (f' : g) + \frac{J(f)}{J(g)}$$

for every $g \in \mathcal{H}^+$ for which $g(x^{-1}) = g(x)$, $\forall x \in X$ and which vanishes outside U.

Theorem 4.2. If J is any θ-Haar integral for G, then J = C I for a suitable $C > 0$.

Proof. Let f_1. $f_o \in \mathcal{H}^+$, where f_o is fixed and K_i be the H-sets of X, such that f_i vanishes outside K_i for i = 1, 0. Again as X is locally-θ-H-closed, θ-CR and θ-T_2, there exists $f_i' \in \mathcal{H}^+$ such that $f_i' = 1$ on K_i for i = 1, 0. Let $\varepsilon > 0$. Then by Lemma 4.1 there exists a θ-open set V_i containing e such that

$$(f_i : g) \leq \varepsilon (f'_i : g) + \frac{J(f_i)}{J(g)} \ldots\ldots\ldots (1)$$

for every $g \in \mathcal{H}^+$ which is such that $g(x^{-1}) = g(x)$, for all $x \in X$ and vanishes outside V_i for i = 1, 0. It is easy to see that

$$J(f_i) \leq (f_i : g) J(g) \ldots \ldots \ldots (2)$$

Let C_i be the H-set support of $f_i' \in \mathcal{H}^+$. Since X is locally θ-H-closed θ-CR, θ-T_2, there exists $f_i'' \in \mathcal{H}^+$ such that $f_i'' = 1$ on C_i. Now we choose $\delta > 0$ so that $\delta (f_i'' : f'_i) < 1$.

Then by Lemma 4.1, there exists θ-open sets U_i containing e such that $(f'_i : g) \leq \delta (f_i'' : g) + \frac{J(f_i')}{J(g)}$

for every $g \in \mathcal{H}^+$ which is such that $g(x^{-1}) = g(x)$ for all $x \in X$ and vanishes outside U_i.

Hence using $(f_i'' : g) \leq (f_i'' : f_i')(f_i' : g)$, we get

$$(f'_i : g)\, J(g)\, [1 - \delta (f_i'' : f'_1)] \leq J(f_i')$$

Since $1 - \delta (f_i'' : f_i')$ is independent of g and is greater than zero we have

$(f_i' : g)\, J(g) \leq \frac{J(f_i')}{1-\delta(f_i'', f_i')} = M_i$ (say), for i = 1, 0. If $U = U_1 \cap U_0$ and $M = \text{Max}(M_1, M_0)$ then

$$(f_i' : g)\, J(g) \leq M \text{ ....(3)}$$

So if $W = V_1 \cap V_0 \cap U$ and if $g \in \mathcal{H}^+$ is such that $g(x^{-1}) = g(x)$ for all $x \in X$ and vanishes outside W, then (1) – (3) holds simultaneously. Therefore

$$\frac{J(f_1)}{\varepsilon(f_0' : g)\, J(g) + J(f_0)} \leq \frac{(f_1 : g)}{(f_0 : g)} \leq \frac{\varepsilon(f_1' : g)\, J(g) + J(f_1)}{J(f_0)}$$

Now from (3),

$$\frac{J(f_1)}{\varepsilon M + J(f_0)} \leq \frac{(f_1 : g)}{(f_0 : g)} \leq \frac{\varepsilon M + J(f_1)}{J(f_0)}.$$

From relation (B) given before Proposition 3.4, we get a sequence $g_n \in \mathcal{H}^+$ such that $g_n(x^{-1}) = g_n(x)$ for all $x \in X$ and g_n vanishes outside W, and $I_{g_n}(f_1)$ converges to $I(f_1)$.

$$\text{So, } \frac{J(f_1)}{\varepsilon M + J(f_0)} \leq I(f_1) \leq \frac{\varepsilon M + J(f_1)}{J(f_0)}$$

If we let $\varepsilon \to 0$ then

$$I(f_1) = \frac{J(f_1)}{J(f_0)}$$

i.e. $J(f_1) = J(f_0)\, I(f_1)$.

and this holds for every $f_1 \in \mathcal{H}^+$.

So $J(f) = J(f_0)\, I(f)$ for every $f \in \mathcal{H}^+$.

i.e. $J(f) = C I(f)$.

REFERENCES

1. S. K. Berbarian, Measure and Integration, the Macmillan Company, 1967.
2. S. V. Fomin, Extensions of topological spaces, Ann, Math, 44(1943), 471-480.
3. S. Ganguly, C. K. Basu and S. Sinha Roy, A new regular measure on a topological space I, Far East J. Math. Sci., Special vol. (1997), Part-1, 35-47.
4. S. Ganguly, C. K. Basu and S. Sinha Roy, Existence of an invariant measure in a generalized topological group, Bull. Cal. Math. Soc. 89 (1997) 131-134.
5. J. R. Porter and R. W. Woods, Extensions and absolutes of Hausdorff spaces, Springer Verlag, 1988.
6. S.Sinha Roy and S. Bandyopadhyay, On θ-Completely regular and locally θ-H-closed spaces, Bull. Cal. Math. Soc., Bull. Cal. Math. Soc. 87 (1995) 19-28.
7. N. V. Veličko, H-closed topological spaces, Mat. Sb. 70(112) (1966) 98-112 Amer. Math. Soc. transl. (2) 78 (1969) 103-118.

Department of Pure Mathematics
University of Calcutta
35, Ballygunge Circular Road
Calcutta 700 019, India