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ON THE CURVATURE TENSOR OF A SEMI-SYMMETRIC SEMI-METRIC CONNECTION

GOBINDA CHANDRA SARKAR

ABSTRACT. A semi-symmetric semi-mertic connection has been introduced by Barua and Mukhopadhyay and the curvature tensor of such a connection has been deduced by them. The nature of the Riemann curvature tensor has been studied after putting some restrictions on the curvature tensor of the semi-symmetric semi-metric connection.

1. INTRODUCTION

A semi-symmetric semi-metric connection $\overline{\nabla}$ on a Riemannian manifold is defined by [1].

$$(\overline{\nabla}_X g)(Y, Z) = 2\alpha(X)\, g(Y, Z) - \alpha(Y)\, g(X, Z) - \alpha(Z)\, g(Y, X) \quad \ldots\ldots (1.1)$$

where g is the Riemann metric on the manifold and α is a differentiable 1-form, called the associated 1-form. The torsion tensor of the connection $\overline{\nabla}$ is given by

$$T(y, z) = \alpha(z)\, y - \alpha(y)\, z \quad \ldots\ldots (1.2)$$

It is known that the connection $\overline{\nabla}$ is of the form [2].

$$\overline{\nabla}_Y Z = \nabla_Y Z - \alpha(Y)\, Z + g(Y, Z)\, A \quad \ldots\ldots (1.3)$$

where ∇ is the Riemann connection and A is the associated vectorfield defined by

$$g(A, X) = \alpha(X) \quad \ldots\ldots (1.4)$$

The curvature tensor $\overline{R}(X, Y)Z$ of $\overline{\nabla}$ is given by

$$\overline{R}(X, Y)Z = R(X, Y)Z - 2d\alpha(X, Y)Z + g(Y, Z)LX - g(X, Z)LY \quad \ldots\ldots (1.5)$$

where $LX = \nabla_X A + \alpha(X)A$ and $R(X, Y)Z$ is the Riemann curvature tensor.

Defining $K(X, Y, Z, U) = g(R(X, Y)Z, U)$ and $\overline{K}(X, Y, Z, U) = g(\overline{R}(X, Y)Z, U)$, we get

$$\overline{K}(X, Y, Z, U) = K(X, Y, Z, U) - 2d\alpha(X, Y)\, g(Z, U) + g(Y, Z)\,\lambda(X, U) - g(X, Z)\,\lambda(Y, U) \quad \ldots\ldots (1.6)$$

where

$$\lambda(X, U) = g(LX, U) = (\nabla_X \alpha)(U) + \alpha(X)\,\alpha(U) \qquad \ldots\ldots (1.7)$$

It can be seen from (1.6) that $\bar{K}(X, Y, Z, U) + \bar{K}(Y, X, Z, U) = 0$. But the other three identities, satisfied by $K(X, Y, Z, U)$ are not, in general, satisfied by $\bar{K}(X, Y, Z, U)$.

By direct calculation follows that—

$$\bar{K}(X, Y, Z, U) + \bar{K}(Y, Z, X, U) + \bar{K}(Z, X, Y, U) = 0$$

if and only if $d\alpha = 0$. Then (1.6) reduces to

$$\bar{K}(X, Y, Z, U) = K(X, Y, Z, U) + g(Y, Z)\,\lambda(X, U) - g(X, Z)\,\lambda(Y, U) \ldots\ldots (1.8)$$

and $\lambda(X, U) = \lambda(U, X)$.

Throughout this paper we have considered $d\alpha = 0$.

2. THE OTHER TWO IDENTITIES

It is known that the Riemann curvature tensor $K(X, Y, Z, U)$ satisfies.

$$K(X, Y, Z, U) + K(X, Y, U, Z) = 0 \text{ and } K(X, Y, Z, U) = K(Z, U, X, Y) \ldots\ldots (2.1)$$

which are, in general, not satisfied by $\bar{K}(X, Y, Z, U)$.

Let $\bar{K}(X, Y, Z, U) + \bar{K}(X, Y, U, Z) = 0$. Then,

$$g(Y, Z)\,\lambda(X,U) - g(X, Z)\,\lambda(Y, U) + g(Y, U)\,\lambda(X, Z) - g(X, U)\,\lambda(Y, Z) = 0$$

Summing for Y and Z, we find

$$\lambda(X, U) = \frac{tr\lambda}{n} g(X, U) \qquad \ldots\ldots (2.2)$$

The same result holds, if $\bar{K}(X, Y, Z, U) = \bar{K}(Z, U, X, Y)$

On the other hand, if (2. 2) holds, then the two identities in (2.1) are satisfied. In this case—

$$\bar{K}(X, Y, Z, U) = K(X, Y, Z, U) + \frac{tr\lambda}{n}\{g(Y, Z)\,g(X, U) - g(X, Z)\,g(Y, U)\} \ldots\ldots (2.3)$$

If now, $\bar{K}(X, Y, Z, U)$ vanishes identicaly, then,

$$K(X, Y, Z, U) = \frac{tr\lambda}{n}\{g(X, Z)\,g(Y, U) - g(X, U)\,g(Y, Z)\} \qquad \ldots\ldots (2.4)$$

Theorem 2.1. If the curvature tensor of a semi-symmetric semi-metric connection vanishes identically, then the manifold is of constant curvature.

3. FIRST SET OF RESTRICTIONS ON $\bar{R}(X, Y)Z$ AND T (Y,Z)

Let the curvature tensor $\bar{R}(X, Y)Z$ and the torsion tensor T (X, Z) of the semi-symmetric semi-metric connection satisfy the relations,

$$\alpha(\bar{R}(X, Y)Z) = O \qquad \ldots\ldots (3.1)$$

and $\bar{R}(X, Y)\, oT = O$ ...... (3.2)

It is known that [3]

$$(\bar{R}(X, Y)T)(U, V) = \bar{R}(X, Y)\,T(U,V) - T(\bar{R}(X, Y)\,U, V) - T(U, \bar{R}(X, Y)V) - (\bar{\nabla}_{T(X,Y)}T)(U,V)$$

In view of (3.1) and (3.2), the above is equivalent to

$$(\bar{\nabla}_{T(X,Y)}T)(U,V) = O \qquad \ldots\ldots (3.3)$$

Now, using (1.3), we get

$$(\bar{\nabla}_{X}T)(U,V) = \{\lambda(X, V) - ag(X, V)\}\,U - \{\lambda(X,U) - ag(X,U)\}\,V \qquad \ldots\ldots (3.4)$$

where $a = g(A, A) = \alpha(A)$. Substituting from (3. 4) in (3. 3) we get

$$\alpha(X)\,[\{\lambda(Y, V) - ag(Y, V)\}\,U - \{\lambda(Y, U) - ag(Y, U)\}\,V]$$
$$= \alpha(Y)\,[\{\lambda(X, V) - ag(X, V)\}\,U - \{\lambda(X, U) - ag(X, U)\}\,V]$$

from which, we get

$$\alpha(X)\,[\{\lambda(Y, V) - ag(Y, V)\} = \alpha(Y) - \{\lambda(X, V) - ag(X, V)\}$$

Putting Y = A, we get form the last equation

$$\alpha(X)\,\{\lambda(A, V) - a\,\alpha(V)\} = a\,\{\lambda(X, V) - ag(X, V)\} \qquad \ldots\ldots (3, 5)$$

Again, putting V = A in (3.5), we get

$$\lambda(X, A) = \frac{\lambda(A, A)}{a}\,\alpha(X) \qquad \ldots\ldots (3.6)$$

Substituting (3.6) in (3.5), we get

$$\lambda(X, V) = \beta\,\alpha(X)\,\alpha(V) + ag(X, V) \qquad \ldots\ldots (3.7)$$

where

$$\beta = \frac{\lambda(A, A)}{a^2} - 1 \qquad \ldots\ldots (3.8)$$

In view of (3. 7), we get from (1. 8)

$$\bar{K}(X, Y, Z, U) = K(X, Y, Z, U) + a\,\{g(Y, Z)\,g(X, U) - g(X, Z)\,g(Y, U)\} + \beta\,\{g(Y, Z)\,\alpha(X)\,\alpha(U) - g(X, Z)\,\alpha(Y)\,\alpha(U)\} \qquad \ldots\ldots (3.9)$$

From (3.8) we find $\beta = O$ if and only if A is a geodesic congruence. If moreover, K (X, Y, Z, U) vanishes identically, then (3.9) reduced to—

$$\bar{K}(X, Y, Z, U) = a\,\{g(Y, Z)\,g(X, U) - g(X, Z)\,g(Y, U)\} \qquad \ldots\ldots (3.10)$$

Theorem 3.1 If the curvature tensor of a semi-symmetric semi-metric connection satisfies (3.1) and (3.2), then it is given by (3.9).

Theorem 3.2 If on a flat manifold the associated vector field of a semi-symmetric semi-metric connection is a geodesic congruence then the manifold is of constant curvature with respect to this connection.

4. SECOND SET OF RESTRICTIONS ON $\bar{R}$ (X, Y) Z AND T (Y, Z)

Let the curvature tensor $\bar{K}$ (X, Y, Z, U) and the torsion tensor T (Y, Z) satisfy,

$$\bar{K}(X, Y, Z, U) = g(Y, U)\{\sigma(X)\alpha(Z) + \sigma(Z)\alpha(X)\} - g(X, U)\{\sigma(Y)\alpha(Z) + \sigma(Z)\alpha(Y)\} \quad \ldots\ldots (4.1)$$

and $(\nabla_X T)(Y, Z) = \sigma(X)\, T(Y, Z)$ ...... (4.2)

where σ is a differentiable 1-form.

From (4.2) we get

$$(\nabla_X \alpha)(Z) = \sigma(X)\alpha(Z) \quad \ldots\ldots (4.3)$$

and $(\nabla_X A) = \sigma(X) A$ ......(4.5)

Therefore

$$\lambda(X, Z) = \sigma(X)\alpha(Z) + \alpha(X)\alpha(Z) \quad \ldots\ldots (4.5)$$

Again—

$\lambda(X, Z) = \lambda(Z, X)$ implies

$$\sigma(X) = t\,\alpha(X),\ t = \frac{\sigma(A)}{\alpha(A)} \quad \ldots\ldots (4.6)$$

Hence $\lambda(X, Z) = (t + 1)\alpha(X)\alpha(Z)$ ......(4.7)

Using (4.7) in (1.8) we get

$$2t\{g(Y, U)\alpha(X)\alpha(Z) - g(X, U)\alpha(Y)\alpha(Z)\}$$
$$= K(X, Y, Z, U) + (t + 1)\{g(Y, Z)\alpha(X)\alpha(U) - g(X, Z)\alpha(Y)\alpha(U)\} \ldots\ldots (4,8)$$

$t = 1$, we find $\sigma(X) = \alpha(X)$ and

$$K(X, Y, Z, U) = 2\{g(Y, U)\alpha(X)\alpha(Z) - g(X, U)\alpha(Y)\alpha(Z) + g(X, Z)\alpha(Y)\alpha(U) - g(Y, Z)\alpha(X)\alpha(U)\} \quad \ldots\ldots (4.9)$$

Hence the manifold is of almost constant curvature [4].

Theorem 4.1. If the curvature tensor of a semi-symmetric semi-metric connection satisfies (4.1) and the torsion tensor is recurrent, then the 1-form of recurrence must be a scalar multiple of the associated 1-form. In this case the associated vectorfield is also recurrent.

Theorem 4.2. If the curvature tensor of a semi-symmeteric semi-metric connection satisfies (4.1) and the torsion tensor is recurrent, the 1-form of recurrence being the associated 1-form, then the manifold is of almost constant curvature.

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Gobinda Chandra Sarkar
Vill. & P.O. Bhangar
Dist. 24-Parganas (South)
Pin 743 502
W.B., India