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ON STRONGLY IRRESOLUTE AND γ-CONTINUOUS FUNCTIONS

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ABSTRACT. Two new classes of functions are introduced and investigated as to their characterizations, properties and interrelations. Relationships with some other allied types of functions are also studied.

1. INTRODUCTION AND PRELIMINARIES

The paper is meant for the initiation to the study of two new types of functions, each of which being independent of continuity. The deliberations are divided into the next two sections, the first one of which deals with the first type viz. strongly irresolute functions while the other type of newly introduced functions termed γ-continuity is taken up in the third section. Some allied classes of functions of the know types that come in the relevance are also treated with regard to their relationships with the newly introduced ones.

Throughout the paper, by X we shall mean a topplogical space (X, τ). A set A in X is said to be semi-open [9] if there exists an open set U such that $U \subset A \subset clU$. The complment of a semi-open set is called semi-closed. We denote the closure and interior of a subset A by clA and intA respectively. Semi-closure [3] of a subset A, which is the intersection of all semi-closed sets containing A, is donoted by sclA. A set A in X is said to be semi-regular [11] if A is semi-open as well as semi-closed. The family of all semi-open (semi-closed, semi-regular) sets of X will be denoted by SO(X) (resp. SC(X), SR(X)) while the collection of all members of SO(X) (SR(X)) each containing a point x of X will be denoted by SO(x) (resp. SR(x)). A point x of X is said to be a semi-θ-adherent point [11] of a subset A of X if $sclU \cap A \neq \emptyset$, for every $U \in SO(x)$. The set of all semi-θ-adherent points of A is called the semi-θ-clesure of A, to be denoted by $[A]_{s\text{-}\theta}$. A is called semi-θ-closed (s-θ-closed, for short) if $A = [A]_{s\text{-}\theta}$. The complement of an s-θ-closed set is called semi-θ-open (s-θ-open, for short). also a set A in X is called regularly open if A = int clA.

It is clear that for any subset A in X, $A \subset sclA \subset [A]_{s\text{-}\theta}$ and therefore every s-θ-open set is semi-open, but converse is not true [12]. In [12] we have shown that the collections of semi-θ-open sets and open sets are non-comparable. Maio and Noiri

[11] proved that every semi-regular set is s-θ-clesed, but there exist s-θ-closed sets which are not semi-regular. The following example substantiates the claim.

Example 1.1. Consider the space (X, τ), where X = {a,b,c} and τ = {∅ ,X, {b,c}, {b}, {c}}. Then SR(X) = {∅,X, {b}, {c}, {a,b}, {a,c}}, and the s-θ-closed sets of X are given by the collection {∅, X, {b}, {c}, {a,b} , {a,c}, {a}}. Here {a} is s-θ-closed but not semi-regular.

Although $[A]_{s\text{-}\theta}$ is the intersection of all s-θ-closed sets containing A [12], the s-θ-closure $[A]_{s\text{-}\theta}$ has been characterized in [12] in terms of semi-regular sets viz. as the intersection of all semi-regular sets containing A.

2. STRONGLY IRRESOLUTE FUNCTIONS

Crossley and Hilderbrand [4] defined a function F : X→Y to be irresolute if for each V∈ SO(Y), f^{-1} (V) is in SO(X). It is easy to see that f : X→Y is irresolute iff for each x ∈ X and each V ∈ SO (f(x)), there is a U ∈ SO(x) such that f(U) ⊂ V. A weaker form of such functions under the terminology quasi-irresolute function was defined by Maio and Noiri [11] as follows. A function f : X→Y is quasi-irresolute if for each x ∈ X and each V ∈ SO(f(x)), there is a U ∈ SO(x) such that f(U) ⊂ sclV. We now introduce a new class of functions as follows.

Definition 2.1. A function f : X→Y is said to be strongly irresolute if for each point x of X and each V ∈ SO(f(x)), there exists a U ∈ SO(f(x) such that f(scl U) ⊂ V.

Remark 2.2. Clearly a strongly irresolute function is irresolute. we give examples below to show that an irresolute function may not be strongly irresolute and that strong irresoluteness does not imply continuity. Since irresoluteness and continuity are independent notions [4], it then following that so are strong irresoluteness and continuity.

Example 2.3. Let X = {a,b,c} and T = {∅, X, {a}}. The identily mapping i : (X,T)→(X, T) is an irresolute map. Now, SO(X) = {∅, X, {a}, {a,b}, {a,c}}. Consider a∈X. Then A = {a} is a semi-open set containing i(a) (= a), but there is no semi-open set U such that i (scl U) ⊂ A, because sclU = X, for amy non-null semi-open set J. Hence i is not strongly irresolute.

Example 2.4. On the set X = {a,b,c}, consider the topology T_1 and T_2 given by T_1 = {∅, X, {a}, {b}, {a,b}} and T_2 = {∅, X, {b,c}}. Then SO(X,T_1) = {∅, X, {a}, {b}, {a,b}, {a,c}. {b,c}} and SO(X, T_2) = T_2. It is easy to check that the identity map from (X, T_1) to (X, T_2) is strongly irresolute but not continuous.

A space X is said to be semi-regular [5] if for each x ∈ X and each semi-closed set F with x ∉ F, there exist disjoint semi-open sets U and V such that x ∈ U and F ⊂ V.

Theorem 2.5. Let X be semi-regular. Then an irresolute finction f : X→Y is strongly irresolute.

Proof : Let x ∈ X and V be a semi-open set containing f(x). Since f is irresolute, there exists a semi-open set U containing x such that f(U) ⊂ V. Since X is semi-regular, there is a semi-open set W containing x such that sclW ⊂ U. Then f (scl W) ⊂ V proving that f is strongly irresolute.

Theorem 2.6. For a function $f : X \to Y$, the following staements are equivalent :
(a) f is strongly irresolute.
(b) $[f^{-1}(B)]_{s\text{-}\theta} \subset f^{-1}(\text{scl}B)$, for any $B \subset Y$.
(c) $f([A]_{s\text{-}\theta}) \subset \text{scl } f(A)$, for any $A \subset X$.
(d) $f^{-1}(B)$ is s-θ-closed in X, for every semi-closed set B in Y.
(e) $f^{-1}(D)$ is s-θ-open in X, for every semi-open set D in Y.

Proof. (a)→(b) : Let B be any set in Y. For any $x \notin f^{-1}(\text{scl}B)$, there exists a semi-open set G in Y containing f(x) such that $G \cap B = \emptyset$. By (a), there is a semi-open set V in X containing x such that $f(\text{scl}V) \subset G$ and hence $f(\text{scl}V) \cap B = \emptyset$. This means that $\text{scl}V \cap f^{-1}(B) = \emptyset$ and hence $x \notin [f^{-1}(B)]_{s\text{-}\theta}$.

(b)→(c) : For any set A in X we have by (b),

$$[A]_{s\text{-}\theta} \subset [f^{-1}(f(A))]_{s\text{-}\theta} \subset f^{-1}(\text{scl } f(A)). \text{ and hence}$$
$$f([A]_{s\text{-}\theta}) \subset ff^{-1}(\text{scl } f(A)) \subset \text{scl } f(A).$$

(c)→(d) : For any semi-closed set B in Y, we get by (c), $f([f^{-1}(B)]_{s\text{-}\theta}) \subset \text{scl } f(f^{-1}(B)) \subset \text{scl}B = B$ so that $[f^{-1}(B)]_{s\text{-}\theta} \subset f^{-1}(B)$. Thus $f^{-1}(B)$ is s-θ-closed.

(d)→(e) : Clear.

(e)→(a) : Let x be a point in X and V be a semi-open set in Y containing f(x). Then $U = f^{-1}(V)$ is semi-θ-open in X, and $x \notin X - f^{-1}(V)$. Since $X - f^{-1}(V)$ is semi-θ-closed, there is a semi-open set U in X such that $x \in U$ and $\text{scl}U \cap (X - f^{-1}(V)) = \emptyset$, i.e., $\text{scl } U \subset f^{-1}(V)$. Hence $f(\text{scl}U) \subset V$ proving that f is strongly irresolute.

The concepts of S-closed and s-closed spaces as introduced in [14] and [11] respectively have been extensively studied by renowned topologists. Genster and Reilly [8] proved that every infinite topological space is embeddable as a closed subspace in a connected S-closed space which is not s-closed.

Definition 2.7. A subset A of a topological space (X,T) is said to be semi-compact [2] (s-closed [11]. S-closed [13]) relative to X or simply an s-compact set (resp. s-set, S-set) iff every cover ϑ of A by semi-open sets of X admits a finite subfamily ϑ_0 such that $A \subset \bigcup_{U \in \vartheta_0} U$ (resp. $A \subset \bigcup_{U \in \vartheta_0} \text{scl } U$, $A \subset \bigcup_{U \in \vartheta_0} \text{cl}U$).

It is clear that for any set A in a space X, A is s-compact →A is an s-set → A is an S-set, it is well-known that the reverse implications are not true, in general.

Theorem 2.8. Let $f : X \to Y$ be strongly irresolute. If A is an s-set in X then f(A) is an s-compact set in Y.

Proof : Let $\{U_\alpha : \alpha \in I\}$ be a cover of f(A) by semi-open sets of Y. For each $x \in A$, there exists $\alpha_x \in I$ such that $f(x) \in U_{\alpha_x}$. Since f is strongly irresolute, there exists a semi-open set V_{α_x} containign x in X such that $f(\text{scl } V_{\alpha_x}) \subset U_{\alpha_x}$. Since $\{V_{\alpha_x} : x \in A\}$ is a cover of A by semi-open sets of X, there exists a finite number of points $x_1, \ldots x_n$ in A such that $A \subset \bigcup_{i=1}^{n} \text{scl } V_{\alpha_{x_i}}$ and hence $f(A) \subset f(\bigcup_{i=1}^{n} \text{scl } V_{\alpha_{x_i}}) = \bigcup_{i=1}^{n} f(\text{scl } V_{\alpha_{x_i}}) \subset \bigcup_{i=1}^{n} \text{scl } U_{\alpha_{x_i}}$. Consequently, f(A) is an s-compact set in Y.

In view of Theorem 2·5 we now deduce :

Corollary 2.9. If a function f is irresolute from a semi-regular space X to another space Y, then the image of an s-set of X is an s-compact set of Y.

3. γ-CONTINUOUS FUNCTIONS

Definition 3.1. A function f : X→Y is said to by γ-continuous at x∈X if for each W ∈ SO (f(x)), there is an open set V containing x such that f(V)⊂sclW. We say that f is γ-continuous on X if f is γ-continuous at each point x of X.

Definition 3.2. Let $\mathscr{F}$ be a filterbase on X. Then ad $\mathscr{F}$ (s-θ-ad $\mathscr{F}$) is defined as the set ∩ {clF : F∈ $\mathscr{F}$} (resp. $\cap\{[F]_{s\text{-}\theta}$: F∈ $\mathscr{F}$ }).

Theorem 3.3. For a function f : X→Y, where X and Y are topological spaces, the following statement are equivalent.

(a) f is γ-continuous.

(b) For each filterbase $\mathscr{F}$ on X, f(ad $\mathscr{F}$)⊂s-θ-ad f($\mathscr{F}$).

(c) f(clA)⊂s-θ-clf(A), for any A ⊂ X.

(d) cl(f^{-1}(A)) ⊂f^{-1}(s-θ-clA), for any A⊂Y.

(e) f^{-1}(F) is closed in X for each s-θ-closed set F of Y.

(f) For each semi-regular set R of Y, f^{-1}(R) is clopen in X.

Proof. (a)→(b) : Let x ∈ ad $\mathscr{F}$ and U any semi-open set in Y containing f(x). Then by (a), there exists an open set V containing x such that f(V)⊂sclU. Since x ∈ ad $\mathscr{F}$. V ∩ F ≠ ∅, for each F ∈ $\mathscr{F}$. Thus sclU ∩ f(F) ≠ ∅, for each F ∈ $\mathscr{F}$. Then f(x) ∈ s-θ-clf(F), for each F ∈ $\mathscr{F}$. and hence f(ad $\mathscr{F}$) ⊂s-θ-ad f ($\mathscr{F}$).

(b)→(c) : Let $\mathscr{F}$ denote the collection of all sets containing A. Then $\mathscr{F}$ is a fillterbase on X. Now by (b), f(ad $\mathscr{F}$)⊂s-θ-ad f ($\mathscr{F}$), but ad $\mathscr{F}$ = clA and s-θ-ad f ($\mathscr{F}$) = s-θ-cl f (A). Hence f(clA) ⊂s-θ-cl f (A).

(c)→(d) : For A ⊂ Y, f^{-1}(A) ⊂ X and hence the proof is immediate.

(d)→(e) : Let A be an s-θ-closed set in Y. Now, x ∈ clf^{-1}(A) implies by virtue of (d), x ∈ f^{-1} (s-θ-clA). Then f(x) ∈s-θ-clA = A, i.e., x ∈ f^{-1}(A). Thus f^{-1}(A) is closed in X.

(e)→(f) : Since every semi-regular set is s-θ-closed as well as s-θ-open, (f) follows at once from (e).

(f)→(a) : For any point x∈X, let W be a semi-open set containing f(x). Then sclW is a semi-regular set containing f(x). By (f), f^{-1}(sclW) is a clopen set in X. Taking V = f^{-1}(sclW) we get f^{-1}(V) ⊂scl W, where x∈V.

Remark 3.4. γ-continuity and continuity are independent notions ; also, so are γ-continuous and irresolute functions. We consider the following examples to this end.

Example 3.5. Let X = {a,b,c} and T = {∅, X, {b}, {c}, {b, c}}. Then SO(X) = {∅, X, {b}, {c}, {b,c}, {a,b}, {a,c}}. The identity mapping i : X→Y is then continuous as well as irresolute, but not γ-continuous.

Example 3.6. Let X = {a,b,c} and T_1 = {∅, X, {c}, {a,b}} and T_2 = {∅, X, {c}, {a, b}, {b}, {b,c}}. Then SO(X, T_1) = T_1 and SO(X, T_2)= T_2. In this cace the identity mapping i, (X, T_1)→(X, T_2) is obviously neither continuous nor irresolute but it is γ-continuous.

Remark 3.7. γ-continuous and strongly irresolute functions are independent of sach other in view if the following examples.

Example 3.8. Let X = {a,b,c} and T = {∅, X, {b}, {c}, {b, c}}. Then SO(X) = {∅, X, {b}, {c}, {b,c}, {a,b}, {a,c}}. The semi-closed and semi-θ-closed sets of (X, T) are given by the collections {∅, X, {a,c}, {a,b}, {a}, {c}, {b}} and {∅, X, {b}, {c}, {a,b}, {a,c}, {a}} respectively. Here i : X→X is obviously strongly irresolute. But i is not γ-continuous ; for, B = {b} is an s-θ-closed set whereas i^{-1} (B) = {b} is not closed.

Example 3.9. In Example 3.6. 1 is γ-continuous but not strongly irresolute.

Remark 3.10. Although each of γ-continuity and strong irresoluteness is independent of continuity, the latter property is achieved as the composition of the former two. That is, if f : X→Y is γ-continuous and g : Y→Z is strongly irresolute then gof : X→Z is continuous.

In the next few theorems we investigate for some properties of γ-continuous functions. Here certain similarities in the behaviours of such functions with those of continuous ones can be noticed, although these two types of functions are independent of each other.

Theorem 3.11. Let f X→Y be a γ-continuous surjection. If X is connected then so is Y.

Proof : If posible, let Y be not connected. Then there exists a non-void proper clopen set V (say) in Y. Obviously V is s-θ-closed as well as s-θ-open. Hence by γ-continuity of f, f^{-1} (V) is a clopen set in X which is non-empty and proper, and this is a contdadiction. Thus Y is connected.

Theorem 3.12. Let f : X→Y be a γ-continuous onto map and X be S-closed. Then Y is s-closed.

Proof : Let $\vartheta = \{U_\alpha : \alpha \in I\}$ be a semi-open cover of Y. Then for each α, f^{-1} (scl U_α) is a closed set, since sclU_α is an-s-θ-closed set in Y. Obviously f^{-1} (scl U_α) is semi-open. Then by S-closedness of X, there exist $\alpha_1 \ldots\ldots \alpha_n \in$ I such that $X \subset \bigcup_{i=1}^{n} f^{-1}(\text{scl } U_{\alpha_i})$ which show that $Y \subset \bigcup_{i=1}^{n} \text{scl } U_{\alpha_i}$ and hence Y is s-closed.

Matheshwri and Prasad [10] defined semi-T_2 axiom in the usual manner by replacing open sets by semi-open sets in the definition of T_2-axiom.

Theorem 3.13. Let f,g : X→Y be two γ-continuous mappings and Y is semi-T_2. Then the set $\{x \in X : f(x) = g(x)\}$ is closed.

Proof : Let $A = \{x \in X : f(x) = g(x)\}$ and $x \notin A$ so that $f(x) \neq g(x)$. As Y is semi-T_2, there exist $U \in$ SO (f (x)) and $V \in$ SO (g (x)) such that $U \cap V = \varnothing$ and hence sclU $\cap$ sclV = ∅. But sclU and sclV being semi-regular sets, f^{-1}(sclU) and g^{-1}(sclV) are clopen sets containing x (using γ-continuity of f and g). Taking $W = f^{-1}(\text{sclU}) \cap g^{-1}(\text{sclV})$. We get an open set W containing x such that $W \cap A = \varnothing$. Therefore A is closed.

Corollary 3.14. Let f, g : X→Y be two γ-continuous mappings and Y be semi-T_2. If f and g coincide on a dense subset of X, then are identical on X.

Theorem 3.15. If Y is semi-T_2 and $f : X \to Y$ is a γ-continuous injection then X is Urysohn.

Proof : Let X_1, X_2, be distinct points of X. Then $f(x_1) \neq f(x_2)$ and hence there exist $U_1 \in SO(f(x_1))$ and $U_2 \in SO(f(x_2))$ such that $U_1 \cap U_2 = \varnothing$. This implies that $sclU_1 \cap sclU_2 = \varnothing$. But $sclU_i$ is semi-regular and hence $f^{-1}(scl\ U_i)$ is clopen, for i = 1, 2. But $f^{-1}(sclU_1) \cap f^{-1}(sclU_2) = \varnothing$. Thus X is Urysohn.

Finally, in order to ascertain the relative positions of the two types of functions introduced here among some of the various types of known functions, we recall the following definition.

Definition 3.16. A function $f : X \to Y$ is

(a) K-continuous [6] (completely irresolute [6]) if inverse image of each semi-open set in Y is open (resp. regularly open) in X.

(b) θ-continuous [7] if for each $x \in X$ and each open set V containing f(x), there is an open set U containing x such that $f(clU) \subset clV$.

(c) Completely continuous [1] if the inverse image of every open subset in Y is a regularly open subset of X.

Remark 3.17. Every K-continuous mapping is γ-continuous, that the converse is false follows from the example below.

Example 3.18. Let $x = \{a,b,c\}$, $\tau_1 = \{\varnothing, X, \{a\}, \{b,c\}\}$, $Y = \{1,m,n\}$ and $\tau_2 = \{\varnothing, \{1\}, Y\}$. We define a function $f : (X, \tau_1) \to (Y, \tau_2)$ by f(a) = 1, f(b) = m and f(c) = n. Then f is γ-comtinuous but not K-continuous.

Remark 3.19. It is easy to see that every γ-continuous function is θ-continuous. But the converse is false, for, in Example 3.5 the identity function i is θ-continuous but not γ-continuous.

Theorem 3.20. Every completely irresolute function is strongly irresolute.

Proof : Let $f : X \to Y$ be completely irresolute. Let V be any semi-closed set in Y. Then $f^{-1}(V)$ is regularly closed, i.e., $f^{-1}(V) = clint\ f^{-1}(V)$. We claim that $f^{-1}(V)$ is s-θ-closed. Let $x \in X = f^{-1}(V)$. Then there exists an open set U_x containing x in X such that $U_x \cap f^{-1}(V) = \varnothing$, i.e., $U_x \cap cl\ int\ f^{-1}(V) = \varnothing$. But $cl\ int\ f^{-1}(V)$ is semi-open and hence $scl\ U_x \cap cl\ int\ f^{-1}(V) = \varnothing$ so that $scl\ U_x \cap f^{-1}(V) = \varnothing$. Then $x \in$ s-θ-cl $f^{-1}(V)$ and hence $f^{-1}(V)$ is s-θ-closed. Consequently, f is strongly irresolute.

Remark 3.21. We notice that a strongly irresolute maping is not completely irresolute. In fact, in Example 3.8, $i : X \to X$ is strongly irresolute, but {a,c} is semi-open without being regularly open.

The implicational aspects of different types of functions discussed so far, are thus finally summed up in the following diagram :

Strong irresolute	$\Rightarrow$	irresolute	$\Rightarrow$	quasi-irresolute
$\Uparrow$		$\Uparrow$		$\Uparrow$
Completely irresolute	$\Rightarrow$	K-continuous	$\Rightarrow$	γ-continuous
$\Downarrow$		$\Downarrow$		$\Downarrow$
Completely continuous	$\Rightarrow$	continuous	$\Rightarrow$	θ-continuous

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