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CONVERGENCE OF THE EIGEN-FUNCTION EXPANSION CORRESPONDING TO A MIXED STURM-LIOUVILLE PROBLEM

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ABSTRACT : The convergence of the eigenfunction expansion of an arbitrary f ε $L^2[a,b]$, (the class of all functions square-integrable on [a,b]), associated with a mixed Sturm-Liouville problem is connected with the convergence of the eigenfunction expansion of f associated with a "suitable" separated Sturm-Liouville problem and thereby the congergence of the said expansion under "Fourier conditions" is established.

Key Words : Mixed and Separated Sturm-Liouville problem (SLP), Resolvent operator, Fourier conditions.

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1. INTRODUCTION

A study of boundary value problem associated with a second order differential equation plays an important role in the theory of differential equations because of its application to the physical problems. The differential equation

$$L[y] = -[p(x)\ y^{(1)}(x)]^{(1)} + q(x)y(x) = \lambda y(x),\ a \le x \le b. \qquad \text{..............(1)}$$

where p and q are real valued functions on [a,b], so that L[y] is well-defined on [a,b] for all sufficiently differentiable functions y, $p(x) \neq 0$ on [a,b] and λ is a complex parameter, is known as the Sturm-Liouville equation.

The boundary value problem consisting of this Sturm-Liouville equation and two boundary conditions, one at each end x = a and x = b is referred to as the separated SLP. In [3] Titchmarsh has discussed thoroughly this separated SLP and has shown that the eigenfunction expansion of an arbitrary function f ε $L^2[a,b]$ with respect to this separated SLP behaves as regards convergence in the same way as the ordinary Fourier series of f i.e. it converges if and only if the ordinary Fourier series converges and it converges to the same limit.

In this paper we deal with the convergence of the eigenfunction expansion of a function $f \varepsilon L^2[a,b]$ with respect to the mixed SLP consisting of the Sturm-Liouville equation (assuming $p(x) = 1$) and the two mixed boundary conditions.

$$\left.\begin{aligned} a_1y(a) + a_2y^{(1)}(a) + a_3y(b) + a_4y^{(1)}(b) = 0, \\ b_1y(a) + b_2y^{(1)}(a) + b_3y(b) + b_4y^{(1)}(b) = 0. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where a_i, b_i ($i = 1, 2, 3, 4$) are real constants.

It is obvious that the a_i 's are not proportional to the b_i's ($i = 1, 2, 3, 4$), otherwise the boundary conditions in (2) will be linearly dependent.

Before we can write down the eigenfunction expansion of an $f \varepsilon L^2[a,b]$ with respect to a SLP, we are to assure that the corresponding SLP is self-adjoint, while a separated SLP is always self-adjoint. A mixed SLP is self-adjoint if and only if the co-efficients a_i, b_i of the boundary conditions (2) satisfy (vide [5])

$$a_1b_2 - a_2b_1 = a_3b_4 - a_4b_3. \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Following Eastham [2], we find that there exists a countable set of values of λ, called the eigenvalues, say $\{\lambda_n\}_n$, (vide [4]) so that the SLP consisting of (1) - (2) [under the restriction(3)], has non-trivial solution, say $\psi_n(\cdot)$, called the eigenfunction, for $\lambda = \lambda_n$.

Given an arbitrary function $f \varepsilon L^2[a,b]$, the series $\sum_n c_n\psi_n(x)$ where $c_n = \int_a^b f(t)\psi_n(t)\,dt$, is known as the eigenfunction expansion of f with regard to the mixed SLP(1) - (2) - (3).

Our aim is to establish the convergence of this eigenfunction expanision $\sum_n c_n\psi_n(x)$ under suitable restrictions. In this direction we prove the following theorem :

Theorem : *Let p and q satisfy the following :*

(a) *p is absolutely continuous on [a,b]* $\forall\, b > 0$ *and* $p(x) > 0$.

(b) *q is continuous and integrable on [a,b]* $\forall\, b > 0$. *Then the eigenfunction expansion of an arbitrary function* $f \varepsilon L^2[a,b]$, *associated with the mixed SLP consisting of (1) – (2) [under the resiriction (3)] behaves, as regards convergence, in the same way as the ordinary Fourier series of f does.*

That is, it converges to $\frac{1}{2}\{f(x + o) + f(x - o)\}$, if, for example f(x) is of bounded variation in the neighbourhood of x. Such conditions on f that ensure the convergence of the Fourier series of f are referred to as "Fourier conditions".

This is achieved by relating this eigenfunction expansion $\sum_n c_n\psi_n(x)$ to the eigenfunction expansion of f, associated with a corresponding "suitable" separated SLP.

It may be pointed out that given a mixed SLP, we are to choose a separated SLP suitably so that our purpose is served. There do exist several such suitable separated SLPs, none of which has any special advantage over the others.

2. RESOLVENT OPERATORS OF THE BOUNDARY VALUE PROBLEM

To avoid cumbersome calculations, we assume that $p(x) = 1$ in (1). The case $p(x) \neq 1$ can be treated with suitable transformation reducing (1) to an equation of the same type but with $p(x) = 1$.

Given any $f \in L^2[a,b]$, we consider the inhomogeneous differential equation

$$-y^{(2)}(x) + q(x)y(x) = \lambda y(x) - f(x). \quad \text{..................(4)}$$

The solution of this differential equation that satisfies the boundary conditions of a SLP is known as the resolvent operator of the said SLP.

Let $\Phi_1(x, \lambda)$ denote the resolvent operator of the mixed SLP (1) – (2) – (3), while $\Phi_1(x, \lambda)$ denotes the resolvent operator of the separated SLP consisting of (1) and the boundary conditions

$$\left.\begin{aligned} a_1 y(a) + a_2 y^{(1)}(a) &= 0, \\ b_3 y(b) + b_4 y^{(1)}(b) &= 0. \end{aligned}\right\} \quad \text{.........................(5)}$$

The importance of this resolvent operator is clear from the following relation :

$$\Phi_1(x, \lambda) = \sum_{n=o}^{\infty} \frac{c_n \psi_n(x)}{\lambda - \lambda_n} \quad \text{...................(6)}$$

established by Titchmarsh [3]–p 16. So, if we consider the integral,

$$I_\Gamma = \frac{1}{2\pi i} \int_\Gamma \Phi_1(x, \lambda)\, d\lambda, \quad \text{...................(7)}$$

where Γ is a large closed contour in the λ–plane, then I_Γ is a finite sum of the Sturm-Liouville expansion $\sum_n c_n \psi_n(x)$.

Hence to prove the convergence of $\sum_n c_n \psi_n(x)$, it is sufficient to prove that I_Γ tends to a finite limit as Γ extends and becomes infinitely large.

If $\phi(x, \lambda)$ and $\chi(x, \lambda)$ are the solutions of (1) satisfying the conditions

$$\left.\begin{aligned} \phi(a, \lambda) &= a_2, \quad \phi^{(1)}(a, \lambda) = -a_1, \\ \chi(b, \lambda) &= b_4, \quad \chi^{(1)}(b, \lambda) = -b_3, \end{aligned}\right\} \quad \text{.........................(8)}$$

Titchmarsh [3] has shown that the Wronskian $W(\phi, \chi)$ is independent of x so that we have

$$\begin{aligned} w(\lambda) &= W(\phi, \chi)(x) \\ &= \phi(x, \lambda)\chi^{(1)}(x, \lambda) - \phi^{(1)}(x, \lambda)\chi(x, \lambda) \\ &= a_1 \chi(a, \lambda) + a_2 \chi^{(1)}(a, \lambda) \\ &= -b_3 \phi(b, \lambda) - b_4 \phi^{(1)}(b, \lambda) \end{aligned} \quad \text{...................(9)}$$

and the resolvent operator $\Phi(x, \lambda)$ is given by

$$\Phi(x, \lambda) = \frac{1}{w(\lambda)} \left[\chi(x, \lambda) \int_a^x \phi(y, \lambda) f(y)\, dy + \phi(x, \lambda) \int_x^b \chi(y, \lambda) f(y)\, dy\right] \quad \text{......(10)}$$

3. THE RELATION BETWEEN THE TWO RESOLVENT OPERATORS

As the resolvent operators $\Phi_1(x, \lambda)$ and $\Phi(x, \lambda)$ satisfy the inhomogeneous equation (4), their difference $\Phi_1(x, \lambda) - \Phi(x, \lambda)$ will satisfy the corresponding homogeneous equation (1) ; so it can be expressed as the linear combination of the two linearly independent solutions $\phi(x, \lambda)$ and $\chi(x, \lambda)$ of (1).

Let $\Phi_1(x, \lambda) - \Phi(x, \lambda) = c_1 \phi(x, \lambda) + c_2 \chi(x, \lambda)$ ...... (11)

where c_1 and c_2 are independent of x.

Using (10) we then have

$$\Phi_1(x, \lambda) = c_1 \phi(x, \lambda) + c_2 \chi(x, \lambda) +$$

$$\frac{1}{W(\lambda)}\left[\chi(x, \lambda)\int_a^x \phi(y, \lambda)(f)(y)\,dy + \phi(x, \lambda)\int_x^b \chi(y, \lambda) f(y)\,dy\right]. \quad ......(12)$$

4. DETERMINATION OF THE CONSTANTS C_1 AND C_2

Since $\Phi_1(x, \lambda)$ satisfies the mixed boundary conditions (2), then

$$a_1\Phi_1(a, \lambda) + a_2\Phi_1^{(1)}(a, \lambda) + a_3\Phi_1(b, \lambda) + a_4\Phi_1^{(1)}(b, \lambda) = 0,$$

$$b_1\Phi_1(a, \lambda) + b_2\Phi_1^{(1)}(a, \lambda) + b_3\Phi_1(b, \lambda) + b_4\Phi_1^{(1)}(b, \lambda) = 0, \quad ......(13)$$

In the sequel, for brevity, we will use the notations

$$\int_a^b \phi(y, \lambda) f(y)\,dy = \{\phi, f\}.$$

$$\int_a^b \chi(y, \lambda) f(y)\,dy = \{\chi, f\}.$$

Substituting for $\Phi_1^{(r)}(a, \lambda)$, $\Phi_1^{(r)}(b, \lambda)$ $(r = 0, 1)$ from (12) in (13) we find the equations for the determination of c_1 and c_2 as

$$c_1\{a_3\phi(b, \lambda) + a_4\phi^{(1)}(b, \lambda)\} + c_2\{w(\lambda) + (a_3b_4 - a_4b_3)\}$$

$$= -\frac{1}{W(\lambda)}(a_3b_4 - a_4b_3)\{\phi, f\},$$

$$c_1\{-(a_1b_2 - a_2b_1) - w(\lambda)\} + c_2\{b_1\chi(a, \lambda) + b_2\chi^{(1)}(a, \lambda)\}$$

$$= -\frac{1}{W(\lambda)}(a_1b_2 - a_2b_1)\{\chi, f\}.$$

$$\text{Let } \Delta(\lambda) = \begin{vmatrix} a_3\phi(b, \lambda) + a_4\phi^{(1)}(b, \lambda) & w(\lambda) + (a_3b_4 - a_4b_3) \\ -(a_1b_2 - a_2b_1) - w(\lambda) & b_1\chi(a, \lambda) + b_2\chi^{(1)}(a, \lambda) \end{vmatrix}$$

By Cramer's rule [writing $\Delta(\lambda)\, w(\lambda) = \Delta w$] we get

$$c_1 = \frac{1}{\Delta W}\left[-(a_3 b_4 - a_4 b_3)\{b_1\, \chi(a, \lambda) + b_2 \chi^{(1)}(a, \lambda)\}\{\phi, f\} -(a_1 b_2 - a_2 b_1)\{w(\lambda) + (a_3 b_4 - a_4 b_3)\{\chi, f\}\right],$$

$$c_2 = \frac{1}{\Delta W}\left[-(a_3 b_4 - a_4 b_3)\{(a_1 b_2 - a_2 b_1) + w(\lambda)\}\{\phi, f\} + (a_1 b_2 - a_2 b_1)\{a_3\, \phi(b, \lambda) + a_4\, \phi^{(1)}(b, \lambda)\}\{\chi, f\}\right]. \qquad \ldots\ldots(15)$$

5. THE EIGENFUNCTION EXPANSION

In order to discuss the convergence of the eigenfunction expansion of $f \varepsilon L^2[a,b]$ with respect to the mixed SLP, we shall consider $\frac{1}{2\pi i}\int_{\Gamma_N} \Phi_1(x, \lambda)\, d\lambda$,

where Γ_N is a closed contour in the λ-plane, which is symmetrical about the real λ-axis. The required eigenfunction expansion is obtained on extending the contour Γ_N indefinitely large.

Putting $\lambda = s^2 = (\sigma + it)^2$, we suppose that the contour Γ_N in the λ-plane corresponds to the contour γ_N in the s-plane, where γ_N consists of the lines

$$\left.\begin{aligned} &\sigma = \sigma_N = (N + \tfrac{1}{2})\,\pi/(b-a), \quad 0 \le |t| \le (N + \tfrac{1}{2})\,\pi/(b-a),\\ &t = \pm\, t_N = \pm(N + \tfrac{1}{2})\,\pi/(b-a),\ 0 \le \sigma \le (N + \tfrac{1}{2})\,\pi/(b-a). \end{aligned}\right\} \qquad \ldots\ldots(16)$$

We shall show that as $N \to \infty$

$$\frac{1}{2\pi i}\int_{\Gamma_N} \Phi_1(x, \lambda)\, d\lambda \to \frac{1}{2}\{f(x+0) + f(x-0)\}, \qquad \ldots\ldots\ldots(17)$$

whenever the Fourier series of f converges to $\frac{1}{2}\{f(x+0) + f(x-0)\}$.

We note that

$$\int_{\Gamma_N} \Phi_1(x, \lambda)\, d\lambda = \int_{\Gamma_N} c_1 \phi(x, \lambda) + c_2 \chi(x, \lambda)\}\, d\lambda + \int_{\Gamma_N} \Phi_1(x, \lambda)\, d\lambda$$

$$= I_1 + \int_{\Gamma_N} \Phi_1(x, \lambda)\, d\lambda \qquad \ldots\ldots\ldots(18)$$

Using the expressions (15) obtained for c_1 and c_2, we get

$$I_1 = \int_{\Gamma_N} \frac{1}{\Delta w}(a_1 b_2 - a_2 b_1)\,[-w(\lambda)\,\chi(x, \lambda) - \{b_1 \chi(a, \lambda) + b_2 \chi^{(1)}(a, \lambda)\}\,\phi(x, \lambda) - (a_1 b_2 - a_2 b_1)\,\chi(x, \lambda)]\,\{\phi, f\} + [-w(\lambda)\,\phi(x, \lambda) + \{a_3 \phi(b, \lambda) + a_4 \phi^{(1)}(b, \lambda)\}\,\chi(x, \lambda) - (a_3 b_4 - a_4 b_3)\,\phi(x, \lambda)]\,\{\chi, f\}.]\, d\lambda$$

$$= \int_{\Gamma_N} \frac{1}{\Delta w} (a_1 b_2 - a_2 b_1) [\xi(x, \lambda) \{\phi, f\} + \eta (x, \lambda) \{\chi, f\}] \, d\lambda$$

$$+ \int_{\Gamma_N} \frac{1}{\Delta w} (a_1 b_2 - a_2 b_1) [\chi (x, \lambda) \{\phi, f\} + \phi (x, \lambda) \{\chi, f\}] \, d\lambda \quad \text{......(19)}$$

where $\xi (x, \lambda) = - \{b_1 \chi (a, \lambda) + b_2 \chi^{(1)} (a, \lambda)\} \phi (x, \lambda) - (a_1 b_2 - a_2 b_1) \chi (x, \lambda)$,
$\eta (x, \lambda) = - (a_3 b_4 - a_4 b_3) \phi (x, \lambda) + \{a_3 \phi (b, \lambda) + a_4 \phi^{(1)} (b, \lambda)\} \chi (x, \lambda)$.

6. ESTIMATES OF $\xi (x, \lambda)$, $\eta (x, \lambda)$, $\phi (x, \lambda)$, $\chi (x, \lambda)$

It is clear that $\xi (x, \lambda)$ and $\eta (x, \lambda)$ are those solutions of (1), which satisfy the conditions

$$\left.\begin{aligned} \xi (a, \lambda) &= - b_2 w (\lambda), & \xi^{(1)} (a, \lambda) &= b_1 w (\lambda), \\ \eta (b, \lambda) &= - a_4 w (\lambda), & \eta^{(1)} (b, \lambda) &= a_3 w (\lambda), \end{aligned}\right\} \quad \text{.........(20)}$$

where $w(\lambda)$ is given by (9).

Hence, by using the method of variation of parameters, we can prove that

$$\xi (x, \lambda) = - \frac{1}{s} b_1 w (\lambda) \sin \{s (x - a)\} - b_2 w (\lambda) \cos \{s (x - a)\}$$

$$+ \frac{1}{s} \int_a^x \sin \{s (x - y)\} q (y) \xi (y, \lambda) \, dy,$$

$$\eta (x, \lambda) = - \frac{1}{s} a_3 w (\lambda) \sin \{s (b - x)\} - a_4 w (\lambda) \cos \{s (b - x)\}$$

$$+ \frac{1}{s} \int_x^b \sin \{s (x - y)\} q (y) \eta (y, \lambda) \, dy.$$

Following Titchmarsh [3] we can then find the estimates of $\xi (x, \lambda)$ and $\eta (x, \lambda)$ as given below

$$\left.\begin{aligned} \xi (x, \lambda) &= - b_2 w (\lambda) \cos \{s (x - a)\} + 0 \{e^{|t|(x-a)}\}, \\ \eta (x, \lambda) &= - a_4 w (\lambda) \cos \{s (b - x)\} + 0 \{e^{|t|(b-x)}\}, \end{aligned}\right\} \quad \text{.........(21)}$$

as $|\lambda| \to \infty$

The estimates of $\phi (x, \lambda)$, $\chi (x, \lambda)$ given below were obtained by Titchmarsh

$$\left.\begin{aligned} \phi (x, \lambda) &= a_2 \cos \{s (x - a)\} + 0 \{|s|^{-1} e^{|t|(x-a)}\}, \\ \chi (x, \lambda) &= b_4 \cos \{s (b - x)\} + 0 \{|s|^{-1} e^{|t|(b-x)}\}, \end{aligned}\right\} \text{ as } |\lambda| \to \infty \quad \text{.........(22)}$$

7. ESTIMATES OF $\int_{\Gamma_N} \{c_1 \phi(x, \lambda) + c_2 \chi(x, \lambda)\} \, d\lambda$.

Using (22) in (14) we can show that as $|\lambda| \to \infty$,

$$\left| \frac{1}{\Delta(\lambda)} \right| = 0 \left(|s|^{-2} e^{-2|t|(b-a)} \right), \qquad \text{.........(23)}$$

$$\{\phi, f\} = \int_a^b \phi(y, \lambda) \, f(y) \, dy = 0 \left(e^{|t|(b-a)} \right), \qquad \text{.........(24)}$$

$$\{\chi, f\} = \int_a^b \chi(y, \lambda) \, f(y) \, dy = 0 \left(e^{|t|(b-a)} \right). \qquad \text{.........(25)}$$

Then using (23) – (25) in (19) we get

$$I_1 = \int_{\Gamma_N} \{c_1 \phi(x, \lambda) + c_2 \chi(x, \lambda)\} \, d\lambda.$$

$$= \int_{\Gamma_N} \frac{1}{\Delta W} (a_1 b_2 - a_2 b_1) \left[\xi(x, \lambda). \{\phi, f\} + \eta(x, \lambda). \{\chi. f\} \right] d\lambda \; +$$

$$\int_{\Gamma_N} \frac{1}{\Delta W} (a_1 b_2 - a_2 b_1) \left[\chi(x, \lambda). \{\phi, f\} + \phi(x, \lambda). \{\chi. f\} \right] d\lambda.$$

$$= 0 \left[\int_{\gamma_N} |s|^{-1} \{ e^{-|t|(x-a)} + e^{-|t|(b-x)} \} \, ds \right], \text{ as } |\lambda| \to \infty. \qquad \text{.........(26)}$$

8. PROOF OF THE THEOREM

Writing $s = \sigma + it$ for the part of the contour Γ_N that corresponds to

$$\sigma = (N + \tfrac{1}{2}) \, \pi / (b - a) = \alpha_N \text{ (say)}, \; -\alpha_N \le t \le \alpha_N,$$

the contribution to I_1 is

$$0 \left[\frac{1}{\alpha_N} \int_{-\alpha_N}^{\alpha_N} \{ e^{-|t|(x-a)} + e^{-|t|(b-x)} \} \, dt \right] = 0 \left(\frac{1}{\alpha_N} \right).$$

Similarly for the part of the contour Γ_N that corresponds to

$$t = \pm (N + \tfrac{1}{2}) \, \pi / (b - a) = \pm \alpha_N \text{ (say)}, \; 0 \le \sigma \le \alpha_N,$$

the contribution to I_1 is

$$0 \left[\frac{1}{\alpha_N} \int_0^{\alpha_N} \{ e^{-|t|(x-a)} + e^{-|t|(b-x)} \} \, d\sigma \right] = 0 \left(\frac{1}{\alpha_N} \right).$$

So, Γ_N being a contour symmetric about the real axis, we get

$$\left.\begin{aligned} I_1 &= \int_{\Gamma_N}\{c_1\,\phi(x,\lambda) + c_2\,\chi(x,\lambda)\}\,d\lambda. \\ &= 0\left(\frac{1}{\alpha_N}\right) = \underset{\sim}{O}(1) \text{ as } N \to \infty . \end{aligned}\right\} \qquad \text{.........(27)}$$

Hence $\int_{\Gamma_N}\Phi_1(x,\lambda)\,d\lambda = \int_{\Gamma_N}\Phi(x,\lambda)\,d\lambda + \underset{\sim}{O}(1)$ as $N \to \infty$ . ...(28)

This proves that the eigenfunction expansion of an arbitrary $f \varepsilon L^2[a, b]$ associated with a mixed SLP is convergent if and only if the eigenfunction expansion of f associated with a suitable separated SLP is convergent.

Following Titchmarsh [3] we know that if f satisties "Fourier condition" then as $N \to \infty$

$$\frac{1}{2\pi i}\int_{\Gamma_N}\Phi_1(x,\lambda)\,d\lambda \to \frac{1}{2}\{f(x+0) + f(x-0)\}.$$

This, in conjunction with (6) and (28), proves that the eigenfunction expansion $\sum_n c_n\psi_n(x)$ of f converges to $\frac{1}{2}\{f(x+0) + f(x-0)\}$ whenever f satisties Fourier conditions. This establishes our theorem.

9. REMARK

Given any mixed SLP we can always select a corresponding separated SLP so that the convergence of the eigenfunction expansion of an arbitrary $f \varepsilon L^2 [a, b]$, corresponding to the mixed SLP is guided by the convergence of the eigenfunction expansion corresponding to the separated SLP as shown in (28). It therefore becomes clear that as far as the convergence of the eigenfunction expansion is required, one need not go into the complication arising due to the mixed boundary conditions ; a suitable set of separated boundary conditions serves the purpose.

In this paper, the separated boundary conditions taken into consideration are given by (5) ; other types of separated boundary conditions might have been chosen but the resulting calculations might be too clumsy. It is obvious that (5) could be replaced by the set of boundary conditions

$$\left.\begin{aligned} b_1\,y(a) + b_2\,y^{(1)}(a) &= 0 \\ a_3\,y(b) + a_4\,y^{(1)}(b) &= 0, \end{aligned}\right\} \qquad \text{...........(20)}$$

without any significant change in calculations.

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