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CHARACTERIZATIONS OF BITOPOLOGICAL QHC SPACES VIA CERTAIN CLASSES OF SETS

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ABSTRACT : In the present article, certain new types of sets are introduced for a bitopological space. This provides a new approach towards the study of bitopological quasi H-closed spaces ; a few characterizations of such a space are established here to justify the contention.

Key Words : ij-θ^c-set, ij-θ^c-set, ij-QHC space, ij-α-open sets, ij-α-closure.

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H-closed and quasi H-closed (QHC) spaces are extremely well known concepts of topology, and the same have so far been studied to a great extent. Extensions of these concepts in bitopological setting have also been investigated by many (e.g. see [3, 5, 7, 9]). The present paper is a continuation of the latter study, and we have introduced here certain types of sets which play key roles in characterizing a bitopological QHC space from an altogether new view point.

Throughout the paper, by a space (X, Q_1, Q_2) or simply by X we shall mean a bitopological space [4] X endowed with two arbitrary topologies Q_1, and Q_2. For a subset A of (X, Q_1, Q_2), Q_i-int A and Q_i-cl A will respectively stand for the interior and closure of A in (X, Q_i), where i = 1, 2. According to Kariofillis [3], a point x in (X, Q_1, Q_2) is said to be in the ij-θ-closure of a subset A of X, written as $x \in$ ij-θ-cl A iff for every Q_i-open set U containing x, $(Q_j\text{-cl } U) \cap A \neq \varnothing$, where and henceforth in every such sentence involving both i and j we assume i, j = 1, 2 and $i \neq j$. A set A in X is called ij-θ-closed iff A = ij-θ-cl A. The complement of an ij-θ-closed set is called an ij-θ-open set. The ij-θ-interior of a set A in X, denoted by ij-θ-int A, consists of those points x of A such that for some Q_i-open set U containing x, $Q_j\text{-cl}U \subset A$. A set A is called ij-θ-open iff A = ij-θ-int A, or iff (X – A) is ij-θ-closed [1, 3]. A Q_i-open set U will be called an ij-regularly open set [12] or simply an ij-roset iff $U = Q_i\text{-int } Q_j\text{-cl } U$ and complements of ij-rosets are called ij-regularly closed sets or simply ij-rc set.

A set A in (X, Q_1, Q_2) is called ij-δ-closed [10] iff for each $x \in (X - A)$, there is a Q_i-open set U containing x such that $(Q_i\text{-int } Q_j\text{-cl } U) \cap A = \varnothing$. The set A is called ij-δ-open [10] iff (X – A) is ij-δ-closed. The class of all ij-δ-open sets in X forms a topology Q_i^s, called the ij-semiregularization topology [8, 12] such that $Q_i^s \subset Q_i$, and the family of all ij-ro sets of (X, Q_1, Q_2) forms a base for Q_i. A property P of a bitopological space (X, Q_1, Q_2) is called a pairwise semiregular property [8] provided (X, Q_1, Q_2) has P iff (X, Q_1^s, Q_2^s) possesses P. We now state a few known results that we shall use in course of the subsequent deliberations.

Theorem 1. [1, 3, 8, 9, 10] For a bitopological space (X, Q_1, Q_2) the following hold :

(a) The collection of ij-θ-open sets in X forms a topology Q_i^θ on X such that $Q_i^\theta \subset Q_i^s \subset Q_i$.

(b) If $A \subset B \subset X$, then ij-θ-int A $\subset$ ij-θ-int B and ij-θ-cl A $\subset$ ij-θ-cl B.

(c) If A ($\subset$ X) is Q_j-closed, then Q_i-int A = Q_i^s int A = ij-θ-int A.

(d) If A is Q_j-open in X, then Q_i-cl A = Q_i^s-cl A = ij-θ-cl A.

We now introduce the following definition.

Definition 2. A set A in a space (X, Q_1, Q_2) is called an

(i) ij-θ°-set iff A = ij-θ-int B, for some $B \subset X$,

(ii) ij-θ^c-set iff A = ij-θ-cl B, for some $B \subset X$.

Note 3. It is easy to see that for any set A in X, ij-θ-cl (X – A) = X – (ij-θ-int A). Thus a set A is an ij-θ°-set iff (X – A) is an ij-θ^c-set.

Proposition 4. For any two sets A and B in a space X,

(a) ij-θ-cl $(A \cup B)$ = (ij-θ-cl A) $\cup$ (ij-θ-cl B)

(b) ij-θ-int $(A \cap B)$ = (ij-θ-int A) $\cap$ (ij-θ-int B).

Proof. (a) Clearly, (ij-θ-cl A) $\cup$ (ij-θ-cl B) $\subset$ ij-θ-cl $(A \cup B)$. Now, $x \notin$ (ij-θ-cl A) $\cup$ (ij-θ-cl B) $\Rightarrow$ there exist Q_i-open sets U, V containing x such that $A \cap Q_j$-cl U = $B \cap Q_j$-cl V = ∅. Then $U \cap V$ is a Q_1-open set containing x such that $(A \cup B) \cap Q_j$-cl $(U \cap V)$ = ∅. Hence $x \notin$ ij-θ-cl (AUB).

(b) Similar to (a) and is omitted.

Corollary 5. The family σ_{ij} = {ij-θ-int A : $A \subset X$} forms a base for some topology $Q_i(\sigma_{ij})$ on X.

Again, from Theorem 1 (c) it follows that

Proposition 6. Every ij-roset is an ij-θ°-set.

It is known [9] that for any subset A of a space (X, Q_1, Q_2), ij-θ-cl A $= \cap\{Q_i$-cl V : $A \subset V \in Q_i\}$. It then follows that ij-θ-int A = U $\{Q_i$-int V : $V \subset A$ and V is Q_j-closed in X$\}$. Hence we obtain :

Theorem 7. Every ij-θ°-set is the union of ij-rosets and hence is ij-δ-open as well as Q_i-open.

It now readily follows in view of corollary 5, Theorem 7 and proposition 6 that

Theorem 8. For a space (X, Q_1, Q_2), $Q_j(\sigma_{ij}) = Q_i^\theta$ and hence σ_{ij} forms an open base for the semiregularization topology Q_i^s on X.

Theorem 9. [5] A space (X, Q_1, Q_2) is said to be ij-QHC iff every Q_i-open cover of X has a finite Q_j-proximate subcover (i.e., a finite subfamily the union of whose members is Q_j-dense in X).

Theorem 10. A space (X, Q_1, Q_2) is ij-QHC iff whenever μ is a cover of X by ji-θ^c-sets such that for each point x of X some member of μ is a Q_i- neighbourhood of x, then μ has a finite subcover.

Proof. Let μ be a cover of an ij-QHC space X by ji-θ^c-sets with the stated property. For each $x \in X$, there exist a $U_x \in \mu$ and a Q_i-open set V_x such that $x \in V_x \subset U_x$. The collection $\{V_x : x \in X\}$ is then a Q_i-open cover of X, and consequently by ij-QHC property of X, there is a finite subset $\{x_1, x_2, ...,x_n\}$ of X such that $X = \bigcup_{k=1}^{n} Q_j\text{-cl}$ $V_{x_k} \subset \bigcup_{k=1}^{n} Q_j\text{-cl } U_{x_k} = \bigcup_{k=1}^{n} U_{x_k}$, since U_{x_k}'s are Q_j-closed sets.

Coversely, let the given condition hold for a space X and μ be a Q_i-open cover of X. For each $x \in X$, there exists $U_x \in \mu$ such that $x \in U_x$. By Theorem 1 (d) we have, $V_x = Q_j\text{- cl } U_x = \text{ji-}\theta\text{-cl } U_x$ and hence V_x is a ji-θ^c-set for each $x \in X$. Then $\{V_x : x \in X\}$ is a cover of X by ji-θ^c-sets with the stipulated property. Thus we obtain,

$X = \bigcup_{k=1}^{n} V_{x_k} = \bigcup_{k=1}^{n} Q_j\text{-cl } U_{x_k}$ for a finite subset $(x_1, x_2,..., x_n\}$ of X, and this proves that X is ij-QHC.

Definition 11. A family $\mathscr{F}$ of sets in a space (X, Q_1, Q_2) is said to possess ij-θ°-FIP iff the ij-θ-interior of every finite intersection of members of $\mathscr{F}$ is non-void.

Theorem 12. A space (X, Q_1, Q_2) is ij-QHC iff every family of ij-θ^c-sets with ji-θ°-FIP has non-null intersection.

Proof. Let X be an ij-QHC space and $\{F_\alpha : \alpha \in \Lambda\}$ be a family of ij-θ^c-sets in X with ji-θ°-FIP. If $\bigcap_{\alpha\in\Lambda} F_\alpha = \varnothing$, then $\mu = \{X - F_\alpha : \alpha \in \Lambda\}$ is a cover of X by ij-θ°-sets and hence μ is also a Q_i-open cover of X. Since X is ij-QHC, we have :

$X = \bigcup_{k=1}^{n} Q_j\text{-cl } \left(X - F_{\alpha_k}\right)$ for a finite subcollection $\{ X - F_{\alpha_1}, X - F_{\alpha_2}, ... X - F_{\alpha_n}\}$ of μ.

Thus $\varnothing = X - \bigcup_{k=1}^{n} Q_j\text{-cl}\left(X - F_{\alpha_k}\right) = \bigcap_{k=1}^{n} Q_j\text{-int } F_{\alpha_k} = \bigcap_{k=1}^{n}$ ji-θ-int F_{α_k} (by therorem 1 (c)) =

ji-θ-int $\left(\bigcap_{k=1}^{n} F_{\alpha k}\right)$ [by Proposition 4 (b)], which contradicts that $\{F_\alpha : \alpha \in \Lambda\}$ has ji-θ°-FIP.

Conversely, let the given condition hold in a space X. We first show that (X, Q_1^S, Q_2^S) is ij-QHC. Since the family of all ij-θ°-sets of X forms a base for Q_i^s, it suffices to show that every cover of X by ij-θ°-sets has a finite Q_j^s-proximate subcover. So let $\mu = \{U_\alpha : \alpha \in \Lambda\}$ be a cover of X by ij-θ°-sets. Then $\{X - U_\alpha : \alpha \in \Lambda\}$ (= $\mathscr{F}$, say) is a family of ij-θ^c-sets with $\cap\mathscr{F} = \varnothing$, so that $\mathscr{F}$ cannot have ji-θ°-FIP. Thus there exists a finite subset $\{X - U_{\alpha_1}, \ldots, X\text{-}U_{\alpha_n}\}$ of $\mathscr{F}$ such that ji-θ-int $\left[\bigcap_{k=1}^{n}\left(X - U_{\alpha_k}\right)\right] = \varnothing$.

Then X = X—ji-θ-int $\left[\bigcap_{k=1}^{n}\left(X - U_{\alpha_k}\right)\right]$ = ji-θ-cl $\left[\bigcup_{k=1}^{n} u_{\alpha_k}\right] = \bigcup_{k=1}^{n}$ ji-θ-cl U_{α_k} (by proposition 4 (a)) $= \bigcup_{k=1}^{n} Q_j^s\text{-cl } U_{\alpha_k}$ (Since U_{α_k}'s are Q_i-open). Hence (X, Q_1^S, Q_2^S) is ij-QHC. Since the property of a space being ij-QHC is known [8] to be a pairwise semiregular property, (X, Q_1, Q_2) is ij-QHC.

Definition 13. Let $\{U_\alpha : \alpha \in D\}$ be a net of ij-θ°-sets in a space X with the directed set (D, ≥) as its domain. A point x of X is said to be an ij-θ-adherent point of the net if for each $\alpha \in D$ and each Q_i-open set V containing x, there exists $\beta \in D$ with $\beta \geq \alpha$ such that $U_\beta \cap Q_j\text{-cl } V \neq \varnothing$.

Theorem 14. A space X is ij-QHC iff every net of non-null ji-θ°-sets has an ij-θ-adherent point.

Proof. Let $\{U_\alpha : \alpha \in D\}$ be a net of non-null ji-θ°-sets in the ij-QHC space X. For each $\alpha \in D$, let F_α = ij-θ-cl $[\cup\{U_\beta : \beta \in D \text{ and } \beta \geq \alpha\}]$. Then $\mathscr{F} = \{F_\alpha : \alpha \in D\}$ is a family of ij-θ^C-sets with ji-θ°-FIP. By Theorem 12 there is an $x \in \bigcap_{\alpha \in D} F_\alpha$. Then for any Q_i-open set V containing x and $\alpha \in D$ $(Q_j\text{ - cl } V) \cap [\cup\{U_\beta : \beta \in D \text{ and } \beta \geq \alpha\}] \neq \varnothing$. Thus there is some $\beta \in D$ with $\beta \geq \alpha$ such that $(Q_j.\text{cl}V) \cap U_\beta \neq \phi$. Hence the net $\{U_\alpha : \alpha \in D\}$ of ji-θ^c-sets in X has an ij-θ-adherent point in X.

Conversely, let $\mathscr{F}$ be a collection of ij-θ^c-sets in X with ji-$\theta°$-FIP. Then the family $\mathscr{F}^*$ of all finite intersections of members of $\mathscr{F}$ becomes a directed set under the relation $\geq$, where $F_1 \geq F_2$ iff $F_1 \subset F_2$ (F_1, $F_2 \in \mathscr{F}^*$). For each $F \in \mathscr{F}^*$, we assign the set ji-θ-int F which is non-null, as $\mathscr{F}$ has ji-$\theta°$-FIP. Then {ji-θ-int F : $F \in (\mathscr{F}^*, \geq)$} is a net of non empty ji-$\theta°$-sets in X. By hypothesis, some point x of X is an ij-θ-adherent point of this net. We only show that $x \in \cap \mathscr{F}$, the rest follows from Theorem 12. Let $F \in \mathscr{F}$ and V be a Q_i-open nbd of x. Since $F \in \mathscr{F}^*$, there is some $G \in \mathscr{F}^*$ with $G \geq F$ (i.e., $G \subset F$) such that (ji-θ-int G) $\cap$ Q_j - cl V $\neq \emptyset$. Then (ji-θ-int F) $\cap$ Q_j - cl V $\neq \emptyset$. Thus $x \in$ ij-θ-cl (ji-θ-int F) = Q_i-cl Q_j-int F (by Theorem 1, since F being an ij-θ^c-set, is Q_i-closed) $\subset$ F. Thus $x \in \cap \mathscr{F}$.

Let us now set the following definition of another class of sets in a bitopological space.

Definition 15. A subset A of a space (X, Q_1, Q_2) is said to be ij–α-open if A $\subset$ Q_i-int Q_j-Cl Q_i-int A.

Remark 16. Clearly, a Q_i-open set is ij-α-open and an ij-α-open set is ij-semiopen (a set A in a space (X, Q_1, Q_2) is called ij-semiopen [2], iff there exists a Q_i-open set U such that U $\subset$A $\subset$ Q_j-cl U, or equivalently, A$\subset$$Q_j$ cl Q_1-int A).

Definition 17. A point x of a space (X, Q_1, Q_2) is said to be an ij-α-adherent point of a set A ($\subset$ X) if every ij-α-open set U containing x interesects A. The set of all ij-α-adherent points of A will be called the ij-α-closure of A, to be denoted by ij-α-cl A.

Lemma 18. For any ji-semiopen set A in a space X, ij-α-cl A = Q_i - cl A.

Proof. Clearly, ij-α-cl A $\subset$$Q_i$-cl A, as Q_i-open sets are ij-α-open. Now, $x \notin$ ij-α-cl A $\Rightarrow$ there is an ij-α-open set V with $x \in V$ such that A $\cap$ V = $\emptyset$ $\Rightarrow$ Q_j-int A $\cap$ Q_i-int V = $\emptyset$ $\Rightarrow$ Q_j-int A $\cap$ Q_j-cl Q_i-int V = $\emptyset$ $\Rightarrow$ Q_j -int A $\cap$$Q_i$-int Q_j-cl Q_i-int V= $\emptyset$ $\Rightarrow$ Q_i-cl Q_j-int A $\cap$ Q_i-int Q_j-cl Q_i-int V =$\emptyset$. Now, $x \in V \subset$ Q_i-int Q_j-cl Q_i-int V = W (say) $\in Q_i$ such that W $\cap$ A $\subset$ W $\cap$ Q_i-cl Q_j-int A (as A is ji-semiopen) = $\emptyset$. Hence $x \notin$ Q_i-cl A. This proves the lemma.

Theorem 19. A space (X, Q_1, Q_2) is ij-QHC iff every cover μ of X by ij-α-open sets has α finite subcollection μ_0 such that X = $\cup${ji-α-cl U : U $\in \mu_0$}.

Proof : Let μ be a cover of an ij-QHC space X by ij-α-open sets. For each U $\in \mu$, Q_i-int U $\subset$ U $\subset$ Q_i-int Q_j-cl Q_i-int U = V(U) (say).

$\Rightarrow$ Q_j-cl Q_i-int U $\subset$ Q_j-cl U $\subset$ Q_j-cl V(U) $\subset$ Q_j-cl Q_i-int U

$\Rightarrow$ Q_j-cl V(U) = Q_j-cl Q_i-int U = Q_j-cl U. ....(1)

By Remark 16, U is ij-semiopen and hence by Lemma 18, ji-α-cl U = Q_j-cl U. Then ji-α-cl U = Q_j-cl V(U) (by (1)) ....(2)

Since U $\subset$ V (U) $\in Q_i$, for all U$\in\mu$, {V(U) : U$\in\mu$} is a Q_i-open cover of X. The ij-QHC property of X then implies that for a finite subcollection μ_0 of μ, X = $\cup$ {Q_j-cl V(U) : U$\in$ μ_0} = $\cup$ {ji-α-cl U : U$\in$ μ_0}. (by (2)).

Conversely, suppose μ is a Q_i-open cover of a space X for which the given condition holds. Then by Remark 16, μ is also a cover of X by ij-α-open sets. Thus for a finite subcollection μ_0 of μ, $X = \cup \{ji\text{-}\alpha\text{-cl } U : U \in \mu_0\}$. By Remark 16 and Lemma 18, $X = \cup \{Q_j\text{-cl } U : U \in \mu_0\}$ proving that X is ij-QHC.

REFERENCES

1. G. K. Banerjee, On pairwise almost strongly θ-continuous mappings, Bull. Cal. Math. Soc. 79 (1987), 314-320.
2. S. Bose, Semi-open sets, semi-continuity and semi-open mappings in bitopological spaces, Bull. Cal. Math. Soc. 73 (1981), 237-246.
3. C. G. Kariofillis, On pairwise almost compactness, Ann. Soc. Sci. Bruxelles 100 (1986), 129-137.
4. J. C. Kelly, Bitopological spaces, Proc. London Math. Soc. 3 (13) (1963), 71-89.
5. M. N. Mukherjee, On pairwise almost compactness and pairwise H-closedness in a bitopological space, Ann. Soc. Sci. Bruxelles 96 (1982), 98-106.
6. M. N. Mukherjee, On pairwise S-closed bitopological spaces, Internat. J. Math. & Math. Sci. 8 (4) (1985), 729-745.
7. M. N. Mukherjee and G. K. Banerjee, A note on QHC and interiorly QHC bitopological spaces, Ann. Stiin. AL. Univ. "AL-I-CUZA" Din IASI 33 (1987), 93-98.
8. M. N. Mukherjee and G. K. Banerjee, Pairwise semiregularization and submaximality of a bitopological space, Ann. Stiin. AL. Univ. "AL-I-CUZA" Din IASI 34 (1988), 19-24.
9. M. N. Mukherjee, G. K. Banerjee and S. Malakar, Bitopological QHC spaces, Indian Jour. Pure Appl. Math. 21 (7) (1990), 639-648.
10. J. N. Nandi, Bitopological near compactness I : Characterizations, Bull. Cal. Math. Soc. 85 (1993), 337-344.
11. S. K. Sen, J. N. Nandi and M. N. Mukherjee, characterizations of some bitopological separation axioms in terms of ij-θ-closure operator, Mat. Vesnik 44 (1992), 75-82.
12. A. R. Singal and S. P. Arya, On pairwise almost regular spaces, Glasnik Mat. 6 (26) (1971), 335-343.

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