Jour. Pure Math., Vol. 14, 1997, pp. 26-34

ON IRRESOLUTE FUZZY MULTIFUNCTIONS

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ABSTRACT : The paper deals with the study of irresolute fuzzy multifunctions. Several characterizations and certain properties of such multifuntions are established.

Key words : Fuzzy multifunctions, fuzzy upper and lower irresolute multifunctions, fuzzy graph multifunctions

1995 AMS subject classification code : Primary 54 A 40
Secondary 54 C 99

1. INTRODUCTION AND PRELIMINARIES

The study of functions taking points of an ordinary topological space X to fuzzy sets [21] in a fuzzy topological space [2] has so far been made by many researchers. Such functions have been termed as fuzzy multifunctions. Papageourgiou [15] took a leading role in this direction. Later on, Mukherjee and Malakar [11] suitably redefined the concepts of lower inverse and lower semicontinuity of such multifunctions in a natural way and obtained certain expected results. The sole objective of all these papers has been to generalize the existing results on some well known functions which are usually treated between two topological spaces.

The class of irresolute functions, a type of functions independent of continuity, was introduced by Crossley and Hildebrand [4], and subsequently studied by many others. In the present paper, an attempt has been made to introduce the idea of irresolute fuzzy multifunctions. In Section 2, we define, characterize and study such a multifunction in its two splitted forms viz. lower and upper irresolute fuzzy multifunctions. In the next section, our intention has been to apply these fuzzy multifunctions to a few situations to determine whether certain topological properties are transfered to the corresponding fuzzy topological properties under such multifunctions.

Throughout the paper, by (X,T) or simply by X we shall mean a topological space in the classical sense and (Y, T_1) or simply Y will stand for a fuzzy topological space (fts, for short) as defined by Chang [2]. A fuzzy point [16] with the singleton support $y \varepsilon Y$ and the value α $(0 < \alpha \leq 1)$ at y will be denoted by y_α. The closure and interior of a set A in X (or a fuzzy set A in Y) will be denoted by clA and int A respectively. The support of a fuzzy set A in Y will be denoted by A_0, i.e. $A_0 = \{y \ \varepsilon \ Y : A(y) \neq 0\}$. For a fuzzy set A in Y, 1-A will stand for the complement of A in Y [21]. For two fuzzy sets A, B in Y, we write $A \leq B$ to mean that $A(y) \leq B(y)$, for each $y \ \varepsilon \ Y$; and write A q B to mean that A is quasicoincident (q-coincident, for short) with B [16]. The negation of the last statement, i.e. when A is not q-coincident with B, is denoted by $A \not{q} B$. A fuzzy set B is called a quasi-neighbourhood (q-nbd, for short) of a fuzzy set A if there exists a fuzzy open set U such that $A \ q \ U \leq B$ [16]. A fuzzy set A (a set A) in an fts Y (in a topological space X) is called fuzzy semiopen [1] (resp. semiopen [7]) if there exists a fuzzy open set (resp. open set) B such that $B \leq A \leq \text{cl } B$. The complements of semiopen (fuzzy semiopen) sets are called semiclosed (fuzzy semiclosed) sets. Union of semiopen (fuzzy semiopen) sets is known [7,1] to be respectively so. Also, clearly every open (fuzzy open, closed, fuzzy closed) set is semiopen (fuzzy semiopen, semiclosed, fuzzy semiclosed). A fuzzy set A in Y is said to be a semi-q-nbd of a fuzzy point y_α [6] if there exists a fuzzy semiopen set V in Y such that $y_\alpha \ q \ V \leq A$. The fuzzy semiclosure of a fuzzy set A in Y, to be denoted by scl A, is the union of all fuzzy points y_α such that every fuzzy semiopen semi-q-nbd of y_α is q-coincident with A [6]. The semiclosure [3] of any set A in X will also be denoted by scl A. It is known [6,3] that semiclosure (fuzzy semiclosure) of a set (fuzzy set) A in X (resp. Y) is the intersection of all semiclosed (fuzzy semiclosed) sets containing A. A (fuzzy) set A is (fuzzy) semiclosed iff A = scl A [6,3]. A (fuzzy) set B is called a (fuzzy) seminbd of a (fuzzy) set A if there is a (fuzzy) semiopen set U containing A and is contained in B. The union of all (fuzzy) semiopen sets contained in a (fuzzy) set A is called the (fuzzy) semi-interior of A, to be denoted by sint A. It is clear that a (fuzzy) set A is (fuzzy) semiopen iff A = sint A.

2. UPPER AND LOWER IRRESOLUTE FUZZY MULTIFUNCTIONS

Definition2.1. [15] Let (X,T) be a topological space in the classical sense and (Y,T_1) be an fts. We say that $F : X \to Y$ is a fuzzy multifunction, if for each $x \ \varepsilon \ X$ F(x) is a fuzzy set in Y. Such A fuzzy multifunction will be called surjective if $F(X) = 1_Y$, where by 1_Y we mean the fuzzy set in Y given by $1_Y(y) = 1$, for all $y \in Y$.

Throughout the paper, by $F : X \to Y$ we shall mean that F is a fuzzy multifunction from a topological space (X,T) to an fts (Y,T_1).

Definition2.2. [11] For a fuzzy multifunction $F : X \to Y$, the upper inverse $F^+(A)$ and the lower inverse $F^-(A)$ of a fuzzy set A in Y are defined as follows :
$F^+(A) = \{ x \ \varepsilon \ X : F(x) \leq A\}$ and $F^-(A) = \{ x \ \varepsilon \ X : F(x) \ q \ A\}$.

The following theorem gives a relation between $F^+(A)$ and $F^-(A)$.

Theorem 2.3. [11] For a fuzzy multifunction $F : X \to Y$, we have $F^-(l - G) = X - F^+(G)$, for any fuzzy set G in Y.

Definition 2.4. A Fuzzy multifunction $F : X \to Y$ is said to be fuzzy

(a) upper irresolute at a point $x_0 \varepsilon X$, if for every fuzzy semiopen set V in Y with $x_0 \varepsilon F^+(V)$ there exists a semiopen set U in X with $x_0 \varepsilon U$ such that $U \subset F^+(V)$,

(b) lower irresolute at a point $x_0 \varepsilon X$, if for every fuzzy semiopen set V in Y with $x_0 \varepsilon F^-(V)$ there exists a semiopen set U in X with $x_0 \varepsilon U$ such that $U \subset F^-(V)$,

(c) upper irresolute (lower irresolute) on X if it is upper irresolute (lower irresolute) at each point of X.

Theorem 2.5. A fuzzy multifunction $F : X \to Y$ is fuzzy upper irresolute iff $F^+(U)$ is semiopen for every fuzzy semiopen set U in Y.

Proof : Let $F : X \to Y$ be a fuzzy upper irresolute multifunction. For a fuzzy semiopen set U in Y, let $x \varepsilon F^+(U)$. Then there is a semiopen set V in X with $x \varepsilon V$ such that $V \subset F^+(V)$. Thus $F^+(U) \subset \text{sint } F^+(U)$ and hence $F^+(U)$ is semiopen in X. Conversely, let $F^+(U)$ be semiopen in X for each fuzzy semiopen set U in Y. Let $x_0 \varepsilon X$ and V be a fuzzy semiopen set in Y with $x_0 \varepsilon F^+(V)$. Now $F^+(V)$ (= U, say) is semiopen in X containing x_0 such that $x_0 \varepsilon U \subset F^+(V)$, proving F to be fuzzy upper irresolute at x_0 and hence on X.

Theorem 2.6. A fuzzy multifunction $F : X \to Y$ is fuzzy lower irresolute iff $F^-(U)$ is semiopen in X for each fuzzy semiopen set U in Y.

Proof : The proof is quite similar to that of Theorem 2.5, and is thus omitted.

Theorem 2.7. A fuzzy multifunction $F : X \to Y$ is fuzzy lower (upper) irresolute iff $F^+(W)$ (resp. $F^-(W)$) is semiclosed in X for every fuzzy semiclosed set W in Y.

Proof : Follows from Theorems 2.3, 2.5 and 2.6.

Lemma 2.8.[13] Let A be an open subset of a topological space (X, T). Then

(a) U ($\subset$ X) is semiopen in X $\Rightarrow U \cap A$ is semiopen in A i. e. in (A, T_A)

(b) U ($\subset$ A) is semiopen in A $\Rightarrow$ U is semiopen in X.

Theorem 2.9. Let $\{U_\alpha : \alpha \varepsilon \Lambda\}$ be an open cover of a topological space X. A fuzzy multifunction $F : X \to Y$ is fuzzy upper (lower) irresolute iff $F/u_\alpha : U_\alpha \to Y$ is fuzzy upper (lower) irresolute for each $\alpha \varepsilon \Lambda$.

Proof : Let $F : X \to Y$ be fuzzy upper (lower) irresolute. Let $x \varepsilon U_\alpha$, for some $\alpha \varepsilon \Lambda$. Suppose V is a fuzzy semiopen set in Y such that $x \varepsilon F^+(V)$ ($x \varepsilon F^-(V)$). Then there is a semiopen set U in X with $x \varepsilon U$ such that $U \subset F^+(V)$ (resp. $U \subset F^-(V)$). By Lemma 2.8. (a), $U \cap U_\alpha$ is semiopen in U_α containing x, and $U \cap U_\alpha \subset (F/u_\alpha)^+ (V)$ (resp. $U \cap U_\alpha \subset (F/u_\alpha)^- (V)$. Hence $F/u_\alpha : U_\alpha \to Y$ is fuzzy upper (lower) irresolute for each $\alpha \varepsilon \Lambda$. Conversely, let $x \varepsilon X$ and V be a fuzzy open set in Y such that $x \varepsilon F^+(V)$ ($x \varepsilon F^-(V)$). Now $x \varepsilon U_\alpha$ for some $\alpha \varepsilon \Lambda$, and $F/u_\alpha : U_\alpha \to Y$ is fuzzy upper (lower) irresolute. So there exists a semiopen set U in U_α such that $F/u_\alpha (U) \leq V$ ($U \subset (F/u_\alpha)^- (V)$). Now by Lemma 2.8 (b), U is semiopen in X and $U \subset U_\alpha$. Thus $F(U) = F/u_\alpha (U) \leq V$ (resp. $F(U) = (F/u_\alpha) (U)\, q\, V$), i. e. $U \subset F^+(V)$ (resp. $U \subset F^-(V)$). Hence F is fuzzy upper (lower) irresolute.

Definition 2.10. Let $F : X \to Y$ be a fuzzy multifunction. The fuzzy multifunction $\text{scl } F : X \to Y$ is defined as $(\text{scl } F)(x) = \text{scl } F(x)$, for each $x \varepsilon X$; and the multifunction $\text{cl}F : X \to Y$ is defined as $(\text{cl } F)(x) = \text{cl } F(x))$, for each $x \varepsilon X$.

Lemma 2.11. If $F : X \to Y$ is a fuzzy multifunction, then $(\text{scl } F)^-(V) = F^-(V)$, for every fuzzy semiopen set V in Y.

Proof : $x \,\varepsilon\, F^-(V) \Rightarrow F(x) \, q \, V \Rightarrow \text{scl } (F(x)) \, q \, V \Rightarrow x \,\varepsilon (\text{scl } F)^-(V)$. Again, $x \,\varepsilon (\text{scl } F)^-(V) \Rightarrow V \, q \text{ scl } (F(x)) \Rightarrow$ there exists $y \,\varepsilon\, V_0 \cap (\text{scl } F(x))_0$ such that $V(y) + (\text{scl } F(x))(y) > 1$. Let $(\text{scl } F(x))(y) = \alpha$. Then $y_\alpha \leq \text{scl } F(x)$ and $\alpha + V(y) > 1$. Then V is a semiopen semi-q-nbd of y_α. Hence $v \, q \, F(x)$, i.e. $x \,\varepsilon\, F^-(v)$.

Theorem 2.12. A fuzzy multifunction $F : X \to Y$ is fuzzy lower irresolute iff $\text{scl } F : X \to Y$ is fuzzy lower irresolute.

Proof : Follows at once from Lemma 2. 11.

Lemma 2.13. If $F : X \to Y$ is a fuzzy multifunction, then $(\text{scl } F)^+(C) = F^+(C)$ for every fuzzy semiclosed set C in Y.

Proof : $x \,\varepsilon\, (\text{scl}F)^+(C) \Rightarrow (\text{scl } F)(x) \leq C \Rightarrow F(x) \leq C \Rightarrow x \,\varepsilon\, F^+(C)$. Again, $x \,\varepsilon\, F^+(C) \Rightarrow F(x) \leq C \Rightarrow \text{scl } F(x) \leq \text{scl } C = C \Rightarrow x \,\varepsilon\, (\text{scl}F)^+(x)$.

Theorem 2.14. A fuzzy multifunction $F : X \to Y$ is fuzzy upper irresolute iff $\text{scl } F : X \to Y$ is upper irresolute.

Proof : Follows immediately from Lemma 2.13.

Theorem 2.15. For A fuzzy multifunction $F : X \to Y$, the following are equivalent:

(a) F is fuzzy lower irresolute.

(b) For each $x \,\varepsilon\, X$ and each fuzzy semi q-nbd V of $F(x)$ in Y, $F^-(v)$ is a semi-nbd of x in X.

(c) For each $p \,\varepsilon\, X$ and each fuzzy semi q-nbd V of $F(p)$ in Y, there is a semiopen set U containing p in X such that $U \subset F^-(V)$.

(d) $F(\text{scl}A) \leq \text{scl } (F(A))$, for any subset A of X.

(e) $\text{scl } (F^+(B)) \subset F^+(\text{scl}B)$, for any fuzzy set B in Y.

(f) $F(\text{int cl}A) \leq \text{scl } F(A)$, for any subset A of X.

Proof : (a) $\Rightarrow$ (b) : Let $x \,\varepsilon\, X$ and V be a fuzzy semi-q-nbd of $F(x)$ in Y, then there exists a fuzzy semiopen set U in Y such that $F(x) \, q \, U \leq V$. Then $x \,\varepsilon\, F^-(U) \subset F^-(V)$. By (a), $F^-(U)$ is a semiopen set in X. Hence $F^-(V)$ is a semi-nbd of x in X.

(b) $\Rightarrow$ (c) : Let $p \,\varepsilon\, X$ and V be a fuzzy semi-q-nbd of $F(p)$ in Y. By (b), $F^-(V)$ is a semi-nbd of p in X. Then there exists a semiopen set U in X such that $p \,\varepsilon\, U \subset F^-(V)$.

(c) $\Rightarrow$ (a) : Let $x \,\varepsilon\, X$ and V be a fuzzy semiopen set in Y with $x \,\varepsilon\, F^-(V)$. Then $f(x) \, q \, V$, i.e. V is a fuzzy semi-q-nbd of $F(x)$ in Y. By (c), there is a semi-open set U containing x in X such that $U \subset F^-(V)$. Hence F is fuzzy lower irresolute.

(a) $\Rightarrow$ (d) : For any $A \subset X$, $\text{scl } F(A)$ is fuzzy semiclosed in Y. By (a), $F^+(\text{scl } F(A))$ is semiclosed in X and clearly contains A. Thus $\text{scl } A \subset F^+(\text{scl } (F(A)))$, i.e., $F(\text{scl } A) \leq F \, F^+(\text{scl } (F(A))) \leq \text{scl } F(A)$.

(d) $\Rightarrow$ (a) : Let V be a fuzzy semiclosed set in Y and let $A = F^+(V)$. By (d), $F(\text{scl } A) \leq \text{scl } (F(A)) = \text{scl } (F \, F^+(V)) \leq \text{scl } V = V$.

Thus $\text{scl } A \subset F^+(V) = A$, i.e. $A = \text{scl } A$. Hence $A = F^+(V)$ is semiclosed in X. Then by Theorem 2.7, F becomes fuzzy lower irresolute.

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(c) upper irresolute (lower irresolute) on X if it is upper irresolute (lower irresolute) at each point of X.

Theorem 2.5. A fuzzy multifunction $F : X \to Y$ is fuzzy upper irresolute iff F^+ (U) is semiopen for every fuzzy semiopen set U in Y.

Proof : Let $F : X \to Y$ be a fuzzy upper irresolute multifunction. For a fuzzy semiopen set U in Y, let $x \varepsilon F^+(U)$. Then there is a semiopen set V in X with $x \varepsilon V$ such that $V \subset F^+(V)$. Thus $F^+(U) \subset \text{sint } F^+(U)$ and hence $F^+(U)$ is semiopen in X. Conversely, let F^+ (U) be semiopen in X for each fuzzy semiopen set U in Y. Let $x_0 \varepsilon X$ and V be a fuzzy semiopen set in Y with $x_0 \varepsilon F^+(V)$. Now $F^+(V)$ (= U, say) is semiopen in X containing x_0 such that $x_0 \varepsilon U \subset F^+(V)$, proving F to be fuzzy upper irresolute at x_0 and hence on X.

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Definition 2.10. Let $F : X \to Y$ be a fuzzy multifunction. The fuzzy multifunction $\text{scl } F : X \to Y$ is defined as $(\text{scl } F)(x) = \text{scl } F(x)$, for each $x \varepsilon X$; and the multifunction $\text{cl}F : X \to Y$ is defined as $(\text{cl } F)(x) = \text{cl } F(x))$, for each $x \varepsilon X$.

Lemma 2.11. If $F : X \to Y$ is a fuzzy multifunction, then $(scl\ F)^-(V) = F^-(V)$, for every fuzzy semiopen set V in Y.

Proof : $x \in F^-(V) \Rightarrow F(x)\ q\ V \Rightarrow scl\ (F(x))\ q\ V \Rightarrow x \in (scl\ F)^-(V)$. Again, $x \in (scl\ F)^-(V) \Rightarrow V\ q\ scl\ (F(x)) \Rightarrow$ there exists $y \in V_0 \cap (scl\ F(x))_0$ such that $V(y) + (scl\ F(x))(y) > 1$. Let $(scl\ F(x))(y) = \alpha$. Then $y_\alpha \leq scl\ F(x)$ and $\alpha + V(y) > 1$. Then V is a semiopen semi-q-nbd of y_α. Hence $v\ q\ F(x)$, i.e. $x \in F^-(v)$.

Theorem 2.12. A fuzzy multifunction $F : X \to Y$ is fuzzy lower irresolute iff $scl\ F : X \to Y$ is fuzzy lower irresolute.

Proof : Follows at once from Lemma 2. 11.

Lemma 2.13. If $F : X \to Y$ is a fuzzy multifunction, then $(scl\ F)^+(C) = F^+(C)$ for every fuzzy semiclosed set C in Y.

Proof : $x \in (sclF)^+(C) \Rightarrow (scl\ F)(x) \leq C \Rightarrow F(x) \leq C \Rightarrow x \in F^+(C)$. Again, $x \in F^+(C) \Rightarrow F(x) \leq C \Rightarrow scl\ F(x) \leq scl\ C = C \Rightarrow x \in (sclF)^+(x)$.

Theorem 2.14. A fuzzy multifunction $F : X \to Y$ is fuzzy upper irresolute iff $scl\ F : X \to Y$ is upper irresolute.

Proof : Follows immediately from Lemma 2.13.

Theorem 2.15. For A fuzzy multifunction $F : X \to Y$, the following are equivalent:

(a) F is fuzzy lower irresolute.

(b) For each $x \in X$ and each fuzzy semi q-nbd V of F(x) in Y, $F^-(v)$ is a semi-nbd of x in X.

(c) For each $p \in X$ and each fuzzy semi q-nbd V of F(p) in Y, there is a semiopen set U containing p in X such that $U \subset F^-(V)$.

(d) $F(sclA) \leq scl\ (F(A))$, for any subset A of X.

(e) $scl\ (F^+(B)) \subset F^+(sclB)$, for any fuzzy set B in Y.

(f) $F(int\ clA) \leq scl\ F(A)$, for any subset A of X.

Proof : (a) $\Rightarrow$ (b) : Let $x \in X$ and V be a fuzzy semi-q-nbd of F (x) in Y, then there exists a fuzzy semiopen set U in Y such that $F(x)\ q\ U \leq V$. Then $x \in F^-(U) \subset F^-(V)$. By (a), $F^-(U)$ is a semiopen set in X. Hence $F^-(V)$ is a semi-nbd of x in X.

(b) $\Rightarrow$ (c) : Let $p \in X$ and V be a fuzzy semi-q-nbd of F (p) in Y. By (b), $F^-(V)$ is a semi-nbd of p in X. Then there exists a semiopen set U in X such that $p \in U \subset F^-(V)$.

(c) $\Rightarrow$ (a) : Let $x \in X$ and V be a fuzzy semiopen set in Y with $x \in F^-(V)$. Then f (x) q V, i.e. V is a fuzzy semi-q-nbd of F (x) in Y. By (c), there is a semi-open set U containing x in X such that $U \subset F^-(V)$. Hence F is fuzzy lower irresolute.

(a) $\Rightarrow$ (d) : For any $A \subset X$, scl F (A) is fuzzy semiclosed in Y. By (a), $F^+(scl\ F(A))$ is semiclosed in X and clearly contains A. Thus $scl\ A \subset F^+(scl\ (F(A)))$, i.e., $F(scl\ A) \leq F\ F^+(scl\ (F(A))) \leq scl\ F(A)$.

(d) $\Rightarrow$ (a) : Let V be a fuzzy semiclosed set in Y and let $A = F^+(V)$. By (d), $F(scl\ A) \leq scl\ (F(A)) = scl\ (F\ F^+(V)) \leq scl\ V = V$.

Thus $scl\ A \subset F^+(V) = A$, i.e. $A = scl\ A$. Hence $A = F^+(V)$ is semiclosed in X. Then by Theorem 2.7, F becomes fuzzy lower irresolute.

(d) $\Rightarrow$ (e) : Let B be a fuzzy set in Y and let A = F^+ (B). Then by (d), F (scl A) $\leq$ scl (F (A)), i.e. F (scl (F^+ (B))) $\leq$ scl (F F^+ (B)) $\leq$ scl B, i.e. scl (F^+ (B)) $\subset$ F^+ (scl B).

(e) $\Rightarrow$ (d) : Let A $\subset$ X. Then B = F (A) being a fuzzy set in Y, by (e) we have scl (F^+ (B)) = scl (F^+ (F(A))) $\subset$ F^+ (scl B) = F^+ (scl F (A)), i.e. scl A $\subset$ F^+ (scl (F (A))). Hence F (scl A) $\leq$ scl (F (A)).

(a) $\Rightarrow$ (f) : Since F is fuzzy lower irresolute, by Theorem 2.7 we have, F^+ (scl (F (A))) to be semiclosed in X, for each A $\subset$ X. Then int cl F^+ (scl (F (A))) $\subset$ F^+ (scl (F (A))). Thus intcl A $\subset$ intcl (F^+ (scl F (A))) $\subset$ F^+ (scl F (A))), i.e. F (intcl A) $\leq$ scl F (A).

(f) $\Rightarrow$ (d) : For any A $\subset$ X, we have sclA = A $\cup$ intclA $\Rightarrow$ F(sclA) = F (A) $\cup$ F (inlcl A) $\leq$ F (A) $\cup$ scl F (A) (by (f)) = scl F (A).

Theorem 2.16. For a fuzzy multifunction F : X $\rightarrow$ Y, the following are equivalent:

(a) F is fuzzy upper irresolute.

(b) For each x ε X and each fuzzy semi-nbd V of F (x) in Y, F^+ (V) is a semi-nbd of x in X.

(c) For each x ε X and each fuzzy semi-nbd V of F (x), there exists a semi-nbd U of x in X such that F (U) $\leq$ V.

(d) For any fuzzy set B in Y, scl (F^- (B)) $\subset$ F^- (scl B).

Proof. (a) $\Rightarrow$ (b) : Let x ε X and V be a fuzzy semi-nbd of F (x) in Y. Then there is a fuzzy semi-open set U in Y such that F (x) $\leq$ U $\leq$ V. As F is fuzzy upper irresolute, by Theorem 2.5, F^+ (U) is a semiopen set in X. Now, x ε F^+ (U) $\subset$ F^+ (V), which shows that F^+ (V) is a semi-nbd of x in X.

(b) $\Rightarrow$ (c) : Obvious.

(c) $\Rightarrow$ (a) : Let x ε X and V be a fuzzy semiopen set in Y such that F (x) $\leq$ V. By (c), there is a semi-nbd U of x in X such that F (U) $\leq$ V. Then there is a semi-open set W in X such that x ε W $\subset$ U, i.e., F (x) $\leq$ F (W) $\leq$ F (U) $\leq$ V. Hence F is fuzzy upper irresolute.

(a) $\Rightarrow$ (d) : For any fuzzy set B in Y, scl B is fuzzy semiclosed and hence by (a), F^- (scl B) is semiclosed in X and obviously contains F^- (B). Thus scl (F^-(B)) $\subset F^-$ (scl B).

(d) $\Rightarrow$ (a) : Let B be fuzzy semiclosed in Y. Then by (d), scl (F^- (B)) $\subset F^-$ (scl B)) = F^- (B). Thus scl (F^- (B))) = F^- (B), i.e. F^- (B) is semiclosed in X. Hence F is fuzzy upper irresolute.

A subset A in a topological space (X, T) can be treated as a fuzzy set in X with characteristic function given by A (x) = 1, if x ε A, and A (x) = 0, if x ε X\A. Thus with this understanding (X, T) becomes a fuzzy topological space too. We then consider the product fts X $\times$ Y in the usual way as defined by Wong [20]. Thus the fuzzy sets of the form U X V with U ε T and V ε T_1 form an open basis for the product fuzzy topology T X T_1, on X $\times$ Y, where for any (x, y) ε X $\times$ Y.

$$(U \times V)(x, y) = \begin{cases} \min (U(x), V(y)) = 0 \text{ if } x \not\varepsilon U \\ V(y), \text{ if } x \,\varepsilon\, U \end{cases}$$

i.e. (U $\times$ V) (x, y) $\leq$ V (y) in this situation.

In [1] Azad set forth a condition under which an fts X is called product related to another fts Y. In the same paper it was proved that if an fts X is product related to another fts Y, then for any fuzzy sets A, B in X and Y respectively, cl $(A \times B)$ = cl $A \times$ cl B.

Definition 2.17 [II] For a fuzzy multifunction $F : X \rightarrow Y$, the fuzzy graph multifunction $G_F : X \rightarrow X \times Y$ of F is defined as $G_F(x)$ = the fuzzy set $x_1 \times F(x)$ of $X \times Y$, where x_1 is the fuzzy set in X, whose value is 1 at $x \in X$ and O at other points of X. We shall write $\{x\} \times F(x)$ for $x_1 \times F(x)$.

Theorem 2·18. When X is product related to Y, a fuzzy multifunction $F : (X, T) \rightarrow (Y, T_1)$ is fuzzy lower irresolute if its fuzzy graph multifunction $G_F : X \rightarrow X \times Y$ is fuzzy lower irresolute.

Proof. Let $G_F : X \rightarrow X \times Y$ be fuzzy lower irresolute. Let $x \in X$ and V be a fuzzy semiopen set in Y such that $x \in F^-(V)$. i.e. $F(x)\ q.\ V$. Then there exists $y \in [F(x)]_0 \cap V_0$ such that $[F(x)](y) + V(y) > 1$. So $[G_F(x)](x, y) + (X \times V)(x,y) = [F(x)](y) + V(y) > 1$. Hence $G_F(x)\ q\ (X \times V)$, where $X \times V$ is clearly a fuzzy semiopen set in $X \times Y$. In fact, V being fuzzy semiopen in Y, $U \leq V \leq$ cl U, for some fuzzy open set U in Y ; then $X \times U$ is a basic open fuzzy set in $X \times Y$ such that $X \times U \leq X \times V \leq$ cl $X \times$ cl U = cl $(X \times U)$, as X is product related to Y. Now, G_F being fuzzy lower irresolute, there is a semiopen set U in X such that $x \in U$, and $G_F(z)\ q\ (X \times V)$, for all $z \in U$. We are only to show that $F(z)\ q\ V$, for all $z \in U$. If possible, let there exist some $z_0 \in U$ such that $F(z_0) \not{q}\ V$. Then for all $y \in Y$, $[F(z_0)](y) + V(y) \leq 1$. Now, for any $(r,s) \in X \times Y$, $[G_F(z_0)](r,s) \leq [F(z_0)](s)$, and $(X \times V)(r, s) = V(s)$. Now, $[G_F(z_0)](r, s) + (X \times V)(r,s) \leq [F(z_0)](s) + V(s) \leq 1$. Thus $G_F(z_0) \not{q}\ (X \times V)$, where $z_0 \in U$. This is a contradiction and hence F is fuzzy lower irresolute.

Theorem 2·19. When X is product related to Y, a fuzzy multifunction $F : (X, T) \rightarrow (Y, T_1)$ is fuzzy upper irresolute if its fuzzy graph multifunction $G_F : X \rightarrow X \times Y$ is fuzzy upper irresolute.

Proof. Let G_F be fuzzy upper irresolute. We shall show that F is fuzzy upper irresolute. Let $x \in X$ and V be a fuzzy semiopen set in Y with $F(x) \leq V$. Then $G_F(x) \leq X \times V$, and $X \times V$ is easily seen to be fuzzy semiopen in $X \times Y$. Then by hypothesis, there is a semiopen set U in X with $x \in U$ such that $G_F(U) \leq X \times V$. Now for any $z \in U$ and for any $Y \in Y$, $[F(z)](y) = [G_F(z)](z,y) \leq (X \times V)(z, y) = V(y)$, i.e. $[F(z)](y) \leq V(y)$, for all $y \in Y$. Thus $F(z) \leq V$, for any $z \in U$, i.e. $F(U) \leq V$. Hence F is fuzzy upper irresolute.

3. APPLICATIONS

It is known [5] that a topological space X is semicompact if every semiopen cover of X has a finite subcover. The definitions of cover and semicompactness in fuzzy setting are now recalled below :

Definition 3.1. [7] A collection ϑ of fuzzy sets in an fts Y is said to be a fuzzy cover of a fuzzy set A in Y if $(\cup \vartheta)(x) = 1$, for each $x \in A_0$; ϑ is called a fuzzy cover of Y if $(\cup \vartheta) = 1_Y$.

Definition 3.2. [8] A fuzzy set A in an fts Y is said be a fuzzy semicompact set if every fuzzy cover ϑ of A fuzzy semiopen sets in Y has a finite subfamily ϑ_0 such that $A \leq \cup \vartheta_0$; if in addition, $A = 1_Y$, then Y is said to be fuzzy semicompact.

Theorem 3.3. Let $F : X \rightarrow Y$ be a surjective fuzzy multifunction and F(x) be a fuzzy semicompact set in Y for each $x \in X$. If F is fuzzy upper irresolute and X is semicompact, then Y is fuzzy semicompact.

Proof : Let $\{A_\alpha : \alpha \in \Lambda\}$ be a fuzzy cover of Y by fuzzy semiopen sets in Y. Now, for each $x \in X$, F(x) is a fuzzy semicompact set in Y, and so there is a finite subset Λ_x of Λ such that $F(x) \leq \cup \{A_\alpha : \alpha \in \Lambda_x\}$. Let $A_x = \{A_\alpha : \alpha \in \Lambda_x\}$. Then $F(x) \leq A_x$, and A_x is fuzzy semiopen in Y. Since F is fuzzy upper irresolute, there exists a semiopen set U_x in X containing x such that $U_x \subset F^+(A_x)$. The family $\{U_x : x \in X\}$ is then a semiopen cover of the semicompact space X. Thus there exist finitely many points $x_1, x_2, \ldots x_n$ in X such that $X = \cup\{U_{x_i} : i = 1,2\ldots, n\}$. As F is surjective, we have

$$1_Y = F(X) = \bigcup_{i=1}^{n} F(U_{x_i}) \leq \bigcup_{i=1}^{n} A_{x_i} \leq \bigcup_{i=1}^{n} \bigcup_{\alpha \in \Lambda_{x_i}} A_\alpha.$$ Hence Y is fuzzy semicompact.

We recall that a subset A of a topological space X is called an S-closed set [14] if every cover of A by semiopen sets of X has a finite proximate subcover ; if in addition, A = X, then X is called an S-closed space [19]. The corresponding definition for an fts runs as follows.

Definition 3.4. [10] A fuzzy set A in an fts Y is said be a fuzzy S-closed set if for every fuzzy cover ϑ of A by fuzzy semiopen sets in Y has a finite proximate subcover, i.e., has a finite subfamily ϑ_0 of ϑ such that $A \leq \cup\{cl\ U : U \in \vartheta_0\}$: if $A = 1_Y$ then the fts Y is called a fuzzy S-closed space.

Lemma 3.5. [9] A topological space X is S-closed iff for every cover ϑ of X by semiopen sets of X has a finite subcollection ϑ_0 such that $X = scl\ (\cup \{U : U \in \vartheta_0\})$.

Theorem 3.6. Let $F : X \rightarrow Y$ be a fuzzy multifunction and F(x) be a fuzzy S-closed set in Y, for each $x \in X$. If F is fuzzy upper irresolute and lower irresolute, and X is S-closed, then Y is fuzzy S-closed.

Proof : Let $\{A_\alpha : \alpha \in \Lambda\}$ be any fuzzy semiopen cover of Y. For each $x \in X$, F(x) being a fuzzy S-closed set in Y, there is a finite subset Λ_x of Λ such that $F(x) \leq \cup \{cl\ A_\alpha : \alpha \in \Lambda_x\} = B_x$(say). Then B_x is a fuzzy semiopen set in Y containing F(x). Since F is fuzzy upper irresolute, $F^+(B_x)$ becomes semiopen in X, by theorem 2.5. Now, $\{F^+(B_x) : x \in X\}$ is a semiopen cover of the S-closed space X. By Lemma 3.5, there exists a finite number of points $x_1, x_2, \ldots x_n$ in X such that $X = scl\ (\bigcup_{i=1}^{n} F^+ (B_{x_i}))$, i. e. $X = scl\ (F^+(\bigcup_{i=1}^{n} (B_{x_i}))$. Now, F being fuzzy lower irresolute, from Theorem 2.15 we have

$$1_Y = F(X) = F\,(\mathrm{scl}\,(F^+(\bigcup_{i=1}^{n} (B_{x_i})))) \le F\,(F^+\,(\mathrm{scl}\,(\bigcup_{i=1}^{n} (B_{x_i})))) \le \mathrm{scl}\,(\bigcup_{i=1}^{n} (B_{x_i}))) = \mathrm{scl}\,(\bigcup_{i=1}^{n} \bigcup_{\alpha \in \wedge_{x_i}} (\mathrm{cl}\,A_\alpha)) = \bigcup_{i=1}^{n} \bigcup_{\alpha \in \wedge_{x_i}} (\mathrm{cl}\,A_\alpha).$$ Hence Y is fuzzy S-closed.

Sinha and Malakar [18] defined a subset A of a topological space (X,T) to be an s-closed set if every cover ϑ of A by semiopen sets of X has a finite subfamily ϑ_o such that $A \subset \cup \{\mathrm{scl}\ U : U \in \vartheta_o\}$.

Theorem 3.7. Let $F : X \to Y$ be a fuzzy multifunction such that F(x) is a fuzzy S-closed set in Y for each $x \in X$. If F is fuzzy upper as well as lower irresolute and a set A in X is an s-closed set, then F(A) is a fuzzy S-closed set in Y.

Proof : Let $\{A_\alpha : \alpha \in \Lambda\}$ be a fuzzy cover of F(A) by fuzzy semiopen sets in Y. For each $x \in A$, F(x) is a fuzzy S-closed set in Y. So there is a finite subset Λ_x of Λ such that $F(x) \le \cup \{\mathrm{cl}\,(A_\alpha) : \alpha \in \Lambda_x\}$. Let $B_x = \cup \{\mathrm{cl}\ A_\alpha : \alpha \in \Lambda_x\}$. Then B_x is fuzzy semiopen in Y and $F(x) \le B_x$. Since F is fuzzy upper irresolute, $F^+\,(B_x)$ is semiopen in X. The family $\{F^+(B_x) : x \in A\}$ is then a cover of the s-closed set A by semiopen sets of X. Consequently, there are finitely many points $x_1, \ldots x_n$ in A such that $A \subset \bigcup_{i=1}^{n} \{\mathrm{scl}\,(F^+\,(B_{x_i}))\}$. Since F is fuzzy lower irresolute, by Theorem 2.15 we have $F(A) \le F\,(\bigcup_{i=1}^{n} \{\mathrm{scl}\,(F^+(B_{x_i}))\}) \le F\,(\mathrm{scl}\,(F^+(\bigcup_{i=1}^{n} B_{x_i}))) \le F\,(F^+(\mathrm{scl}\,(\bigcup_{i=1}^{n} (B_{x_i})))) \le \mathrm{scl}\,(\bigcup_{i=1}^{n} (B_{x_i}) = \mathrm{scl}\,(\bigcup_{i=1}^{n} \bigcup_{\alpha \in \wedge_{x_i}} \mathrm{cl}\ A_\alpha) = \bigcup_{i=1}^{n} \bigcup_{\alpha \in \wedge_{x_i}} \mathrm{cl}\,(A_\alpha)$. Hence F(A) is a fuzzy S-closed set in Y.

ACKNOWLEDGEMENT

We wish to record our deep sense of gratitude to our Supervisor Dr. M. N. Mukherjee of the Department of Pure Mathematics, University of Calcutta for his kind help and guidance throughout the preparation of this paper.

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